Exam Papers Practice

Pure Mathematics: Proof

Name: ______________________

Class: ______________________

Date: ______________________

Time:

Total marks available:

Total marks achieved: ______

Edexcel IAL AS and A Levels Mathematics
Topic : Pure Mathematics
Sub Topic : Proof
Type : Mark Schemes

INTERNATIONAL ADVANCED LEVEL
EDEXCEL INTERNATIONAL GCSE
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First examination June

To be used by all students preparing for Edexcel IAL AS and A Levels Mathematics

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Mark Scheme

Q1.

Question	Scheme	Marks
	A solution based around a table of results <table><tr><td>n</td><td>n^2</td><td>n^2+2</td><td></td></tr><tr><td>**1**</td><td>**1**</td><td>**3**</td><td>**Odd**</td></tr><tr><td>**2**</td><td>**4**</td><td>**6**</td><td>**Even**</td></tr><tr><td>**3**</td><td>**9**</td><td>**11**</td><td>**Odd**</td></tr><tr><td>**4**</td><td>**16**</td><td>**18**</td><td>**Even**</td></tr><tr><td>**5**</td><td>**25**</td><td>**27**</td><td>**Odd**</td></tr><tr><td>**6**</td><td>**36**</td><td>**38**</td><td>**Even**</td></tr></table>	
	When n is odd, n^2 is odd (odd $\times$ odd = odd) so n^2+2 is also odd	M1
	So for all odd numbers n, n^2+2 is also odd and so cannot be divisible by 4 (as all numbers in the 4 times table are even)	A1
	When n is even, n^2 is even **and a multiple** of 4, so n^2+2 cannot be a multiple of 4	M1
	Fully correct and exhaustive proof. Award for both of the cases above plus a final statement "So for all n, n^2+2 cannot be divisible by 4"	A1*
		(4)

	Alternative - (algebraic) proof	
	If n is even, $n = 2k$, so $\frac{n^2+2}{4} = \frac{(2k)^2+2}{4} = \frac{4k^2+2}{4} = k^2 + \frac{1}{2}$	M1
	If n is odd, $n = 2k+1$, so $\frac{n^2+2}{4} = \frac{(2k+1)^2+2}{4} = \frac{4k^2+4k+3}{4} = k^2+k+\frac{3}{4}$	M1
	For a partial explanation stating that • either of $k^2+\frac{1}{2}$ or $k^2+k+\frac{3}{4}$ are not a whole numbers. • with some valid reason stating why this means that n^2+2 is not a multiple of 4.	A1
	Full proof with no errors or omissions. This must include • The conjecture • Correct notation and algebra for both even and odd numbers • A full explanation stating why, for all n, n^2+2 is not divisible by 4	A1*
		(4)
		(4 marks)

Q2.

Question	Scheme			Marks
	Assumption: there exists positive real numbers a, b such that $a+b<2\sqrt{ab}$			B1
	Method 1	**Method 2**		
	$a+b-2\sqrt{ab}<0$ $(\sqrt{a}-\sqrt{b})^2<0$	$(a+b)^2=(2\sqrt{ab})^2$ $a^2+2ab+b^2<4ab$ $a^2-2ab+b^2<0$ $(a-b)^2<0$	A complete method for creating $(f(a,b))^2<0$	M1A1
	This is a contradiction, therefore If a, b are positive real numbers, then $a+b\geqslant 2\sqrt{ab}$			A1
				(4)
				(4 marks)

Notes:

B1: As this is proof by contradiction, the candidate is required to start their proof by assuming that the contrary. That is "if a, b are positive real numbers, then $a+b\geqslant 2\sqrt{ab}$" is true.

Accept, as a minimum, there exists a and b such that $a+b<2\sqrt{ab}$

M1: For starting with $a+b<2\sqrt{ab}$ and proceeding to either $(\sqrt{a}-\sqrt{b})^2<0$ or $(a-b)^2<0$

A1: All algebra is required to be correct. Do not accept, for instance, $(a+b)^2=2\sqrt{ab}^2$ even when followed by correct lines.

A1: A fully correct proof by contradiction. It must include a statement that $(a-b)^2<0$ is a contradiction so if a, b are positive real numbers, then $a+b\geqslant 2\sqrt{ab}$

Q3.

Question Number	Scheme	Marks
(i)	$(x-4)^2 \geqslant 2x-9 \Rightarrow x^2-10x+25 \ldots 0$	M1
	$\Rightarrow (x-5)^2 \ldots 0$	A1
	Explains that "square numbers are greater than or equal to zero" hence (as $x \in \mathbb{R}$), $\Rightarrow (x-4)^2 \geqslant 2x-9$ *	A1*
		(3)
(ii)	Shows that it is not true for a value of n	
	Eg. When $n=3$, $2^n+1=8+1=9$ × Not prime	B1
		(1)
		(4 marks)

Notes

(i) A proof starting with the given statement

M1 Attempts to expand $(x-4)^2$ and work from form $(x-4)^2 \ldots 2x-9$ to form a 3TQ on one side of an equation or an inequality

A1 Achieves both $x^2-10x+25$ and $(x-5)^2$. Allow $(x-5)^2$ written as $(x-5)(x-5)$

A1* For a correct proof. Eg

"square numbers are **greater than or equal** to zero", hence (as $x \in \mathbb{R}$), $(x-5)^2 \geqslant 0$

$\Rightarrow (x-4)^2 \geqslant 2x-9$

This requires (1) Correct algebra throughout, (2) a correct explanation concerning square numbers and (3) a reference back to the original statement

Answers via b^2-4ac are unlikely to be correct. Whilst it is true that there is only one root and therefore it touches the x-axis, it does not show that it is always positive. The explanation could involve a sketch of $y=(x-5)^2$ but it must be accurate with a minimum on the +ve x axis with some statement alluding to why this shows $(x-5)^2 \geqslant 0$

Approaches via odd and even numbers will usually not score anything. They would need to proceed using the main scheme via $(2m-4)^2 \geqslant 4m-9$ and $(2m-1-4)^2 \geqslant 2(2m-1)-9$

Alt to (i) via contradiction

Proof by contradiction is acceptable and marks in a similar way

M1 For setting up the contradiction
'Assume that there is an x such that $(x-4)^2 < 2x-9 \Rightarrow x^2-10x+25 \ldots 0$

A1 $\Rightarrow (x-5)^2 \ldots 0$ or $(x-5)(x-5) \ldots 0$

A1* This is not true as square numbers are always greater than or equal to 0, hence $(x-4)^2 \geqslant 2x-9$

Alt to part (i) States $(x-5)^2 \geqslant 0$

$\Rightarrow x^2-10x+25 \geqslant 0$

$\Rightarrow x^2-8x-16 \geqslant 2x-9$

$\Rightarrow (x-4)^2 \geqslant 2x-9$

M1	States $(x-5)^2 \geqslant 0$ and attempts to expand. There is no explanation required here
A1	Rearranges to reach $x^2-8x-16 \geqslant 2x-9$
A1*	Reaches the given answer $(x-4)^2 \geqslant 2x-9$ with no errors
(ii)	
B1	Shows that it is not true for a value of n This requires a calculation (and value found) with a minimal statement that it is not true Eg. '$2^6+1=65$ which is not prime' or '$2^5+1=33$ ×' Condone sloppily expressed proofs. Eg. ' $2^7+1=\frac{129}{3}=43$ which is not prime' Condone implied proofs where candidates write $2^5+1=33$ which has a factor of 11 If there are lots of calculations mark positively. Only one value is required to be found (with the relevant statement) to score the B1 The calculation cannot be incorrect. Eg. $2^3+1=10$ which is not prime

Q4.

Question Number	Scheme	Marks
	Assume that there exists a number m such that when m^3 is even, m is odd	B1
	If m is odd then $m = 2p+1$ (where p is an integer) and $m^3 = (2p+1)^3 = ...$	M1
	$= 8p^3 + 12p^2 + 6p + 1$	A1
	$2\times(4p^3 + 6p^2 + 3p) + 1$ is odd and hence we have a contradiction so if n^3 is even, then n is even.	A1
		(4)
		(4 marks)

B1: For setting up the contradiction.

Eg Assume that there exists a number m such that when m^3 is even, m is odd
Condone a contra-positive statement here
"Assume that there exists a number m such that when m^3 is even, m is not even"
As a minimum accept "assume if m^3 is even then m is odd."
Condone the other way around " assume if n is odd then n^3 is even"

M1: Attempts to cube an odd number. Accept an attempt at $(2p+1)^3$, $(2p-1)^3$

Look for $(2p+1)^3 = ...p^3.....$

A1: $(2p+1)^3 = 8p^3 + 12p^2 + 6p + 1$ or simplified equivalent such as $2\times(4p^3 + 6p^2 + 3p) + 1$.

For $(2p-1)^3 = 8p^3 - 12p^2 + 6p - 1$ or equivalent such as $2\times(4p^3 - 6p^2 + 3p - 1) + 1$

A1: For a fully correct proof. Requires correct calculations with reason and conclusion

E.g. 1 Correct calculations $(2p+1)^3 = 8p^3 + 12p^2 + 6p + 1 =$
Reason (even +1) = odd
Conclusion "hence we have a contradiction, so if n^3 is even, then n is even."

E.g. 2 Correct calculations $(2p+1)^3 = 8p^3 + 12p^2 + 6p + 1$
Reason $= 2\times(4p^3 + 6p^2 + 3p) + 1$ = odd
Conclusion "this is contradiction, so proven."

E.g. 3 Correct calculations $(2p-1)^3 = 8p^3 - 12p^2 + 6p - 1$
Reason = $8p^3 - 12p^2 + 6p$ is even so $8p^3 - 12p^2 + 6p - 1$ is odd

Conclusion: So if n^3 is even then n must be even

Note that B0 M1 A1 A1 is possible

Q5.

Question Number	Scheme	Marks
(i)	(As $x \geqslant 0$ so $\sqrt{3x}$ exists and) $\left(\sqrt{3x}-1\right)^2 \geqslant 0$	M1
	Hence $3x - 2\sqrt{3x} + 1 \geqslant 0$	M1
	$\Rightarrow 3x + 1 \geqslant 2\sqrt{3x}$ *	A1*
		(3)
Alt 1	$3x+1 \geqslant 2\sqrt{3x} \Leftrightarrow (3x+1)^2 \geqslant 12x \Leftrightarrow 9x^2 - 6x + 1 \geqslant 0$	M1
	$9x^2 - 6x + 1 \geqslant 0 \Leftrightarrow (3x-1)^2 \geqslant 0$	M1
	Square numbers are greater than or equal to zero so $(3x-1)^2 \geq 0$ is true hence $3x+1 \geqslant 2\sqrt{3x}$ *	A1*
		(3)
Alt 2	If $3x+1 < 2\sqrt{3x}$ then $3x - 2\sqrt{3x} + 1 < 0$	M1
	So $(...\sqrt{3x} \pm ...)^2 < 0$ or $(\sqrt{x} \pm ...)^2 < 0$	M1
	But $(\sqrt{3x}-1)^2 \geq 0$ for all $x \geqslant 0$ so $3x+1 \geqslant 2\sqrt{3x}$	A1
(ii)	Shows that it is not true for three consecutive prime numbers Eg $5+7+11=23$ which is not divisible by 5 (so not true)	B1
		(1)
		(4 marks)

(i)

M1 Uses that ($x \geqslant 0$ and) squares are non-negative to set up a suitable equation.

M1 Squares to achieve 3 terms.

A1 Rearranges correctly to the given result.

Alt 1: (Backwards proof)

M1 Starting with the given statement, attempts to square both sides, expand $(3x+1)^2$ (three terms required) and collect terms on one side of the inequality.

M1 Attempts to complete the square/factorise the expression to achieve a perfect square or (following error) $(..x+..)^2+..$ (inequality not needed here).
Alternatively, uses other valid method (such as discriminant is zero) to show the resulting expression is non-negative. Finding a single solution alone is not sufficient, there must be a reason why the expression is never negative. (E.g. for $9x^2-6x+1=0$ discriminant is $(-6)^2-4\times9=0$, so single root hence as positive quadratic, $9x^2-6x+1\geqslant 0$)

A1* Achieves $(3x-1)^2\geq 0$ and a statement such as since this latter equation is true (as squares are never negative) hence $3x+1\geq 2\sqrt{3x}$. Alternatively, they may achieve an inequality such as $(\sqrt{3x}-1)^2\geq 0$ and conclude in a similar way.
Note that the proof should really have two way implications at each stage, but allow full credit for proofs that do not show this.

Alt 2: (Contradiction type proof)

M1 **Starts with the negation** of the statement and gathers terms on ones side of the equation.

M1 Attempts to factorise (oe method) the resulting expression to achieve a perfect square.

A1* Achieves $(\sqrt{3x}-1)^2<0$ or alternative suitable expression and concludes as this is false the starting assumption was false, hence the original statement is true.
NB They may substitute for $\sqrt{3x}=y$ or similar, which is fine and can score full marks if correctly reasoned.

NB: there are variations between the methods, but in general look for setting up a correct equation/gathering and identifying the underlying quadratic for the first M, attempting to factorise to a perfect square/complete the square to a non-negative expression or expand a perfect square (as appropriate) for the second M, and with all steps correct and suitable conclusion for the final A. Simply rearranging the equation alone is not sufficient for the first M.

(ii)
B1 Provides a correct counter example, such as 5 + 7 + 11 = 23, or 11 + 13 + 17 = 41, and gives a conclusion. Must see the sum evaluated correctly. The conclusion may be minimal e.g. "hence shown", "not divisible by 5" etc.
The conclusion should come from a correct example, so if they conclude from an incorrect example, it will be B0. But if they have multiple examples, at least one of which is a correct one, and a generic conclusion, B1
NB 1+2+3 is **not** a valid counter example as 1 is not prime.

Q6.

Question	Scheme	Marks
(i)	At least three of: For $p = 2$: $2^3 + 2 = 8 + 2 = 10$; For $p = 3$: $3^3 + 3 = 27 + 3 = 30$ For $p = 5$: $5^3 + 5 = 125 + 5 = 130$; For $p = 7$: $7^3 + 7 = 343 + 7 = 350$	M1
	Each case gives a multiple of 10. As 2,3,5 and 7 are the only single digit primes, the result has been proved for all single digit primes.	A1
		(2)
(ii)	$(n+1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1$	M1A1
	$= 3(n^2 + n) + 1$ which is one more than a multiple of 3, so is not divisible by 3 for any $n \in \mathbb{N}$	A1
		(3)
		(5 marks)

Notes:

(i)

M1: Checks result for at least three of the four single digit primes (2,3,5 and 7) – attaining a multiple of 10 is enough, no need to see the product. Allow if there are slips.

A1: All four cases correctly checked, with minimal conclusion that the result is true. Ignore checks on non-prime values such as $p = 8$ which gives $8^3 + 8 = 520$, but award A0 if the case $p = 1$, leads to a conclusion that the result is not true.

(ii)

Note this appears as MMA on ePEN but is being marked as MAA.

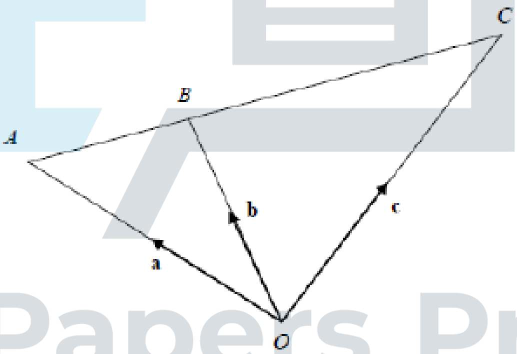
M1: Expands to a four term cubic (may have incorrect coefficients) and then cancels the n^3 terms

A1: Correct quadratic in n.

A1: Correct explanation and conclusion given. For the explanation accept e.g factors out 3 from the relevant terms to achieve a form $3 \times (n^2 + n) + 1$ which is one more than a multiple of 3, or explains each term other than 1 is divisible by 3

Q7.

Question Number	Scheme	Marks
(i)	Attempts two of $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$ either way around	M1
	Attempts $\mathbf{c} - \mathbf{b} = 2 \times (\mathbf{b} - \mathbf{a})$ oe such as $\mathbf{c} - \mathbf{a} = 3 \times (\mathbf{b} - \mathbf{a})$	dM1
	$\Rightarrow \mathbf{c} = 3\mathbf{b} - 2\mathbf{a}$ *	A1 *
		(3)
(ii)	Assume that there exists a number n that isn't a multiple of 3 yet n^2 is a multiple of 3	B1
	If n is not a multiple of 3 then $m = 3p+1$ or $m = 3p+2$ $(p \in \mathbb{N})$ giving	
	$m^2 = (3p+1)^2 = 9p^2 + 6p + 1$	M1
	Or $m^2 = (3p+2)^2 = 9p^2 + 12p + 4 = 3(3p^2 + 4p + 1) + 1$	M1 A1
	$(3p+1)^2 = 9p^2 + 6p + 1\left(= 3(3p^2 + 2p) + 1\right)$ is one more than a multiple of 3	
	$(3p+2)^2 = 9p^2 + 12p + 4$ is not a multiple of 3 as 3 does not divide into 4 (exactly)	
	Hence if n is a multiple of 3 then n^2 is a multiple of 3	A1
		(5)
		(8 marks)



(i)

M1: Attempts any two of $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$.

Condone the wrong way around but it must be subtraction.

Allow marked in the correct place on a diagram

dM1: Uses the given information.

Accept $\overrightarrow{AB} = \frac{1}{3}\overrightarrow{AC}$, $\overrightarrow{BC} = 2 \times \overrightarrow{AB}$, $\overrightarrow{BC} = \frac{2}{3} \times \overrightarrow{AC}$ etc condoning slips as in previous M1.

A1*: Fully correct work inc bracketing leading to the given answer $\mathbf{c} = 3\mathbf{b} - 2\mathbf{a}$

Expect to see the brackets multiplied out. So $\mathbf{c} - \mathbf{b} = 2 \times (\mathbf{b} - \mathbf{a}) \Rightarrow \mathbf{c} - \mathbf{b} = 2\mathbf{b} - 2\mathbf{a} \Rightarrow \mathbf{c} = 3\mathbf{b} - 2\mathbf{a}$ is fine.

(ii)

B1: For setting up the contradiction.

Eg Assume that there exists a number n that isn't a multiple of 3, yet n^2 is a multiple of 3

As a minimum accept something like " define a number n such that n is not a multiple of 3 but n^2 is"

M1: States that $m = 3p+1$ or $m = 3p+2$ and attempts to square.

Alternatives exist such as $m = 3p+1$ or $m = 3p-1$

Using modulo 3 arithmetic it would be $1 \to 1$ and $2 \to 4 = 1$

M1: States that $m = 3p+1$ AND $m = 3p+2$ and attempts to square o.e.

A1: Achieves forms that can be argued as to why they are NOT a multiple of 3

E.g. $m^2 = (3p+1)^2 = 3\left(3p^2 + 2p\right) + 1$ or even $9p^2 + 6p + 1$

and $m^2 = (3p+2)^2 = 3\left(3p^2 + 4p + 1\right) + 1$ or even $9p^2 + 12p + 4$

A1: Correct proof which requires

- Correct calculations
- Correct reasons. E.g. $9p^2 + 12p + 4$ is not a multiple of 3 as 4 is not a multiple of 3

  There are many ways to argue these. E.g $m^2 = (3p+1)^2 = 3\left(3p^2 + 2p\right) + 1$ is sufficient as long as followed (or preceded by) "not a multiple of 3"
- Minimal conclusion such as ✓. Note that B0 M1 M1 M1 A1 is possible

Q8.

Question Number	Scheme	Marks
(a)	(If x and y are positive) $\left(\sqrt{x} - \sqrt{y}\right)^2 \ldots 0 \Rightarrow x - \ldots\sqrt{xy} + y \ldots 0$	M1
	$\Rightarrow x - 2\sqrt{xy} + y \ldots 0$	A1
	$\Rightarrow \dfrac{x+y}{2} \ldots \sqrt{xy}$	A1*
		(3)
(b)	States for example when $x = -8, y = -2,\ \dfrac{x+y}{2} = -5,\ \sqrt{xy} = 4$ so	B1
	$\dfrac{x+y}{2} \leqslant \sqrt{xy}$	(1)
		(4 marks)

(a)

M1: Sets up a correct inequality and attempts to expand $\left(\sqrt{x}-\sqrt{y}\right)^2$ leading to three terms.

A1: Correct expanded equation.

A1: Rearranges to the required equation with no errors seen.

If working in reverse allow the first M and A (if steps correct) but require also a minimal conclusion for the final A.

(b)

B1: Gives a suitable example with both sides evaluated correctly and a minimal conclusion.

There is no need to refer to x and y in the conclusion, so long as it has been shown the required inequality does not hold. E.g. $\frac{-8-2}{2}=-5 \not\geqslant 4=\sqrt{16}=\sqrt{-8\times-2}$ QED is fine.

A common response most likely to score 2 out of 3 marks

Question Number	Scheme	Marks
(a) Alt 1	$\frac{x+y}{2}\geqslant\sqrt{xy}\Rightarrow\frac{(x+y)^2}{4}\geqslant xy\Rightarrow\frac{x^2+\ldots xy+y^2}{4}\geqslant xy$	M1
	$\Rightarrow x^2-2xy+y^2\geqslant 0\Rightarrow(x-y)^2\geqslant 0$	A1
	States both of the following o.e • $(x-y)^2\geqslant 0$ as it is a square number • so $\frac{x+y}{2}\geqslant\sqrt{xy}$ is true	A1*
		(3)

M1: Assumes $\frac{x+y}{2}\geqslant\sqrt{xy}$ true and attempts to square obtaining at least three terms.

A1: Correct expansion and rearranges the inequality correctly to factorise to a perfect square.

A1: A complete conclusion given.

Question Number	Scheme	Marks
(a) Alt 2	States $(x-y)^2 \dots 0 \Rightarrow x^2 - 2xy + y^2 \dots 0$	M1
	Rearranges $\Rightarrow x^2 + 2xy + y^2 \dots 4xy \Rightarrow (x+y)^2 \dots 4xy$	A1
	States that as x, y positive, so $x+y>0$ (and $xy>0$) $\Rightarrow (x+y) \dots \sqrt{4xy} \Rightarrow \dfrac{x+y}{2} \dots \sqrt{xy}$	A1*
		(3)

M1: Sets up an inequality using an appropriate perfect square and expands to at least three terms.

A1: Makes a correct rearrangement and factors the left hand side to produce the equation shown.

A1: Makes a full conclusion justifying why the square root gives $x+y$.

Q9.

Question Number	Scheme	Notes	Marks
(i)	E.g. $p=7 \Rightarrow 2p+1=15$ Which is not a prime number (so the statement is not true)	Identifies a counter example and makes a conclusion/shows it is not prime.	B1
			(1)
(ii)	n odd $\Rightarrow n=2k+1$ $\Rightarrow 5n^2+n+12=5(2k+1)^2+2k+1+12$ **or** n even $\Rightarrow n=2k$ $\Rightarrow 5n^2+n+12=5(2k)^2+2k+12$	Starts the proof by considering n odd **or** n even and substituting into the expression (see notes for logical approach)	M1
	n odd $\Rightarrow n=2k+1$ $\Rightarrow 5n^2+n+12=5(2k+1)^2+2k+1+12$ **and** n even $\Rightarrow n=2k$ $\Rightarrow 5n^2+n+12=5(2k)^2+2k+12$	Considers n odd **and** n even (as above)	M1
	n odd : $20k^2+22k+18$ which is even and n even : $20k^2+2k+12$ which is even	Attempts both with **at least one correct** expression that is stated to be even	A1
	n odd : $2(10k^2+11k+9)$ and n even : $2(10k^2+k+6)$ These are both even so $5n^2+n+12$ must be even for all integers n	Fully correct proof that considers both n odd and n even, shows the resulting expressions are even and makes a suitable conclusion	A1
			(4)
			Total 5

Q10.

Question	Scheme		Marks
(a)	$(S =)a + (a + d) + \ldots + [a+(n-1)d]$	B1: requires at least 3 terms, must include first and last terms, an adjacent term and dots!	B1
	$(S =)[a+(n-1)d] + \ldots\ldots + a$	M1: for reversing series (dots needed)	M1
	$2S = [2a+(n-1)d] + \ldots\ldots + [2a+(n-1)d]$	dM1: for adding, must have $2S$ and be a genuine attempt. Either line is sufficient. Dependent on 1st M1.	dM1
	$2S = n[2a+(n-1)d]$ $S = \frac{n}{2}[2a+(n-1)d]$ cso	(NB −Allow first 3 marks for use of l for last term but as given for final mark)	A1
			(4)
(b)	$600 = 200 + (N-1)20 \Rightarrow N = \ldots$	Use of 600 with a **correct** formula in an attempt to find N.	M1
	$N = 21$	cso	A1
			(2)

(c)	**Look for an AP first:**		
	$S = \frac{21}{2}(2 \times 200 + 20 \times 20)$ or $\frac{21}{2}(200 + 600)$	M1: Use of correct sum formula with their integer $n = N$ or $N-1$ from part (b) where $3 < N < 52$ and $a = 200$ and $d = 20$.	M1A1
	$S = \frac{20}{2}(2 \times 200 + 19 \times 20)$ or $\frac{20}{2}(200 + 580)$ (= 8400 or 7800)	M1: Use of correct sum formula with their integer $n = N$ or $N-1$ from part (b) where $3 < N < 52$ and $a = 200$ and $d = 20$.	
	Then for the constant terms:		
	$600 \times (52 - \text{"}N\text{"})$ (= 18600)	M1: $600 \times k$ where k is an integer and $3 < k < 52$	M1
		A1: A correct un-simplified follow through expression with their k consistent with n so that $n + k = 52$	A1ft
	So total is 27000	cao	A1
	There are no marks in (c) for just finding S_{52}		
			(5)
			(11 marks)

Q11.

Question Number	Scheme	Marks
	States the largest odd number and an odd number that is greater E.g. odd number n and $n+2$	M1
	Fully correct proof including • the assumption: there exists a greatest odd number "n" • a correct statement that their second odd number is greater than their assumed greatest odd number • a minimal conclusion " this is a contradiction, hence proven" You can ignore any spurious information e.g. $n > 0$, $n+2 > 0$ etc.	A1*
		(2)
		(2 marks)

M1: For starting the proof by stating an odd number and a larger odd number.

Examples of an allowable start are

- **odd number** "n" with "$n+2$"
- **odd number** "n" with "n^2"
- "$2k+1$" with "$2k+3$"
- "$2k+1$" with "$(2k+1)^3$"
- "$2k+1$" with "$2k+1+2k$"

Note that stating $n = 2k$, even when accompanied by the statement that "n" is odd is M0

A1*: A fully correct proof using contradiction

This must consist of

1) An assumption E.g. "(Assume that) there exists a greatest odd number n"
"Let "$2k+1$" be the greatest odd number"

2) A minimal statement showing their second number is greater than the first,
E.g. If "n" is odd and "$n+2$" is greater than n

If "n" is odd and $n^2 > n$

$2k+3 > 2k+1$

$2k+2k+1 > 2k+1$

Any algebra (e.g. expansions) must be correct. So $(2k+1)^2 = 4k^2 + 2k + 1$ would be A0

3) A minimal conclusion which could be
"hence there is no greatest odd number", "hence proven", or simply ✓

Q12.

Question Number	Scheme	Marks
(a)	Writes $2\left(4p^3+6p^2+3p\right)+1$ which is odd	B1
		(1)
(b)	Assumption:	
	E.g. States that there exists integers p and q such that $\sqrt[3]{2}=\frac{p}{q}$ (where $\frac{p}{q}$ is in its simplest form) and then cubes to get $2=\frac{p^3}{q^3}$	M1
	$2=\frac{p^3}{q^3}\Rightarrow p^3=2q^3$ and concludes that p^3 is even so therefore p is even	A1
	If p is even then it can be written $p=2m$ so $(2m)^3=2q^3$	M1
	States that $q^3=4m^3$ and concludes that q^3 is even so therefore q is even	A1
	This contradicts our initial statement, as if they both have a factor of 2 it means that $\frac{p}{q}$ is not in its simplest form, so $\sqrt[3]{2}$ is irrational *	A1*
		(5)
		(6 marks)

(a)
B1: See scheme. Requires correct reason/algebra and a statement of the expression being odd.
Allow even + even + even +1 = odd. Allow $2p\left(4p^2+6p+3\right)+1=\text{odd}$

(b)
M1: Sets up the contradiction AND cubes. Condone the omission of the fact that $\frac{p}{q}$ is in its simplest form for this mark. Condone as a minimum $\sqrt[3]{2}=\frac{p}{q}$ followed by $2=\frac{p^3}{q^3}$ o.e.

A1: States that $p^3=2q^3$ and concludes both that p^3 is even so therefore p is even.
Accept other equivalent statements to even such as "multiple of 2"
Condone poor explanations so long as they state that both p^3 and p are even

M1: Writes $p=2m$ so $(2m)^3=2q^3$ and then attempts to find $q^3=\ldots$

A1: States that $q^3=4m^3$ and concludes that both q^3 is even so therefore q is even
Accept other equivalent statements to even such as "multiple of 2"
Condone poor explanations so long as they state that both q^3 and q are even

A1*: Completely correct proof and conclusion with no missing statements.
To score this final mark the statements **now** need to be the correct way around .
E.g. q^3 is even so therefore q is even
It requires $\frac{p}{q}$ to be in simplest form (or equivalent such as no common factor) in the initial assumption.

Q13.

Question	Scheme	Marks
	Assume the sequence is geometric	B1
	So $(r=)\dfrac{1+2k}{k}=\dfrac{3+3k}{1+2k}$	M1
	$\Rightarrow (1+2k)^2=k(3+3k)\Rightarrow k^2+k+1=0$	A1
	But $k^2+k+1=\left(k+\dfrac{1}{2}\right)^2-\dfrac{1}{4}+1\geqslant\dfrac{3}{4}>0$ (since $\left(k+\dfrac{1}{2}\right)^2\geqslant 0$ for all (real) k)	dM1
	This is a contradiction and hence the original assumption is not true. The sequence is not geometric.	A1
		(5)
		(5 marks)

Notes:

(a)
B1: States an appropriate assumption to set up the contradiction.
M1: Uses the assumption to set up an equation in k only.

Allow equivalent work e.g. $kr=1+2k,\ kr^2=3+3k\Rightarrow 3+3k=k\left(\dfrac{1+2k}{k}\right)^2$

Allow use of $\neq$ for $=$ e.g. $\sqrt{\dfrac{3+3k}{k}}\neq\dfrac{1+2k}{k}$

This may be implied by e.g. $\dfrac{1+2k}{k}$ is not the same as $\dfrac{3+3k}{1+2k}$.

A1: Reaches a correct quadratic equation in k, need not be all on one side, but terms in k and k^2 should be collected. Allow use of $\neq$ for $=$ e.g. $k^2+k+1\neq 0$

dM1: Completes the square, considers the discriminant or other valid means used to reach a point where a contradiction can be deduced. E.g. as scheme, or $b^2-4ac=1^2-4(1)(1)=-3<0$ may be used. Accept use of calculator to give roots $k=\dfrac{-1\pm i\sqrt{3}}{2}$ so k is not real, which contradicts k being a member of the real sequence.
Depends on the previous M.

A1: Correct work leading to a contradiction with deduction of a contradiction made and conclusion given. **This mark is available even if B0 is given at the start. So 01111 is possible.**

If they are using the discriminant or calculator route then there is no need to mention "real" as long as they conclude that e.g. the geometric sequence is not possible. This can score both the dM1 and A1.

Q14.

Question Number	Scheme	Marks
	Solves one of $(3x-y)=25$ and $(x+y)=1$ or $(3x-y)=5$ and $(x+y)=5$	M1
	Correct solution of one. Either $\left.\begin{matrix} 3x-y=25 \\ x+y=1 \end{matrix}\right\} \Rightarrow 4x=26 \Rightarrow x=6.5, (y=-5.5)$ Or $\left.\begin{matrix} 3x-y=5 \\ x+y=5 \end{matrix}\right\} \Rightarrow 4x=10 \Rightarrow x=2.5, (y=2.5)$	A1
	Solves both equations	dM1
	Both solved correctly with a minimal reason given for the contradiction e.g "not integers" with conclusion "hence there are no integers x and y such that $3x^2+2xy-y^2=25$"	A1
		(4)

Notes

M1: Attempts to solve one of the two possible cases.

Take as a minimum, one correct pair of equations followed by a value for x or a value for y

A1: Correctly solves one of the two possible cases.

To solve you need only find a value for x or y. Once a correct value is found you can ISW

dM1: Attempts to solve both possible cases

A1: Correctly solves the two possible cases and makes a concluding argument.

To score this mark (i) **all** calculations must be correct.

(ii) reason(s) for the contradiction must be written down. Allow "not integers", "×"

and (iii) gives a concluding statement must be given. Allow for example " hence proven"

Ignore any possible cases which would give rise to negative numbers but satisfy $(3x-y)(x+y)=25$

E.g $(3x-y)=-5, (x+y)=-5$

Withhold the final mark only if they include cases which do not satisfy $(3x-y)(x+y)=25$

E.g $(3x-y)=20, (x+y)=5$

Q15.

Question	Scheme	Marks
	For question many variations on the proof are possible. Below is a general outline with some examples, which cover many cases. If you see an approach you do not know how to score, consult your team leader. **M1:** Will be scored for setting up an algebraic statement in terms of a variable (integer) k or any other variable aside n that engages with divisibility by 4 in some way and can lead to a contradiction and is scored at the point you can see each of these elements. A formal statement of the assumption is not required at this stage. **A1:** Scored for a correct statement from which it is possible to draw a contradiction. **dM1;** For making a complete argument that leads to a (full) contradiction of the initial statement, though may be allowed if there are minor gaps or omissions. **A1:** Correct and complete work with contradiction drawn and conclusion made. There must have been a statement of assumption at the start for which to draw the contradiction, though it may not be technicality a correct assumption as long as a relevant assumption has been made. E.g. Accept "Assume n^2-2 is divisible be 4 for all n"	
	(Assume that there is an n with n^2-2 is divisible by 4 so) $n^2-2=4k$	M1
	then $n^2=4k+2=2(2k+1)$ (so is even)	A1
	Hence n^2 is even so n (=2m) is even hence n^2 is a multiple of 4 As n^2 is a multiple of 4 then $n^2-2=4m^2-2=2(2m^2-1)$ cannot be a multiple of 4 (as $2m$ — 1 is odd) so there is a contradiction.	dM1
	So the original assumption has been shown false. Hence "n^2-2 is never divisible by 4" is true for all n *	A1*
		(4)
		(4 marks)

Notes		
M1: Sets up an algebraic statement in terms of a variable (integer) k or any other variable aside n that engages with divisibility by 4 in some way and can lead to a contradiction. No need for explicit statement of assumption - accept if just a suitable equation is set up. In this case supposing divisibility by 4 by stating $n^2-2=4k$ **A1:** Reaches $n^2=2(2k+1)$ **dM1:** For a complete argument that leads to a contradiction. See scheme. Allow if minor details are omitted as long as the overall argument is clear. Accept explanations such as "as n^2 is even then n is even hence n^2 is a multiple of 4 so n^2-2 cannot be a multiple of 4 (as 4 does not divide 2)" **A1*:** Draws the contradiction to their initial assumption and concludes the statement is true for all n. There must have been a clear assumption at the start that is contradicted, and all working must have been correct. For the assumption be generous with the technicality as long as a relevant assumption has been made. E.g. Accept "Assume n^2-2 is divisible be 4 for all n"		
Alt 1	(Assume that n^2-2 is divisible by 4 for some n,) so $\frac{n^2-2}{4}$ is an integer. Then if n is even $n=2m$ (m integer) so $\frac{n^2-2}{4}=\frac{(2m)^2-2}{4}$ (oe with odd)	M1
	$=m^2-\frac{1}{2}$ (which is not an integer)	A1

	Since m^2 is an integer, $m^2-\frac{1}{2}$ is not, hence n cannot be even, but if n is odd then $\frac{n^2-2}{4}=\frac{(2m+1)^2-2}{4}=m^2+m-\frac{1}{4}$, which is again not an integer (since m^2+m is)	dM1
	Hence there is a contradiction (as n cannot be an integer) Hence "n^2-2 is never divisible by 4" is true for all n *	A1*
		(4)
		(4 marks)

Notes

M1: Sets up an algebraic statement in terms of a variable (integer) m or any other variable aside n that engages with divisibility by 4 in some way and can lead to a contradiction. No need for explicit statement of assumption - accept if just a suitable equation is set up. In this Alt, consider case use of $n = 2m$ or $n = 2m + 1$ in $\frac{n^2-2}{4}$ is sufficient

A1: Reaches $m^2-\frac{1}{2}$ for n even or $m^2+m-\frac{1}{4}$ for n odd.

dM1: For a complete argument that leads to a contradiction in both cases. See scheme. Allow if minor details are omitted as long as the overall argument is clear.

A1*: Draws the contradiction to their initial assumption and concludes the statement is true for all n. There must have been a clear assumption at the start that is contradicted, and all working must have been correct. For the assumption be generous with the technicality as long as a relevant assumption has been made. E.g. Accept "Assume n^2-2 is divisible be 4 for all n"

Alt 2	(Assume that n^2-2 is divisible by 4) $\Rightarrow n^2-2=4k$	M1
	$\Rightarrow n^2=4k+2 \Rightarrow n=2\sqrt{k+\frac{1}{2}}$ or $n=\sqrt{2}\sqrt{2k+1}$	A1
	So for some integer m $\sqrt{k+\frac{1}{2}}=\frac{m}{2}\Rightarrow 2k+1=\frac{m^2}{2}$ but m^2 is odd if m is odd so $\frac{m^2}{2}$ not an integer, or m^2 is a multiple of 4 if m even, so odd=even or $2k + 1$ is odd, so does not have a factor 2 to combine with the $\sqrt{2}$ outside, hence n must be irrational	dM1
	Hence we have a contradiction. So "n^2-2 is never divisible by 4" is true for all n *	A1*
		(4)
		(4 marks)

Notes

M1: Sets up an algebraic statement in terms of a variable (integer) k or any other variable aside n that engages with divisibility by 4 in some way and can lead to a contradiction. No need for explicit statement of assumption - accept if just a suitable equation is set up. In this case supposing divisibility by 4 by stating $n^2-2=4k$

A1: Reaches $n=2\sqrt{k+\frac{1}{2}}$ or $n=\sqrt{2}\sqrt{2k+1}$

dM1: For a complete argument that leads to a contradiction. See scheme. Allow if minor details are omitted as long as the overall argument is clear. Must be a valid attempt to show that $2\sqrt{k+\frac{1}{2}}$ / $\sqrt{2}\sqrt{2k+1}$ is not an integer, and this method is a hard route.

A1*: Draws the contradiction to their initial assumption and concludes the statement is true for all n. There must have been a clear assumption at the start that is contradicted, and all working must have been correct. For the assumption be generous with the technicality as long as a relevant assumption has been made. E.g. Accept "Assume n^2-2 is divisible be 4 for all n"

Alt 3	(Assume that n^2-2 is divisible by 4) then for n even we have (for some integer m) $n^2-2=4m^2-2$ **or** for n odd $n^2-2=4(m^2+m)-1$	M1
	$4m^2-2$ or $4(m^2+m)-1$	A1
	Since 4 divides n^2-2 and $4m^2$ thus for n even, 4 must divide 2, a contradiction, so n cannot be even, and also 4 divides $4(m^2+m)$ so for n odd, 4 divides 1, also a contradiction.	dM1
	Hence we have a contradiction for both cases (and as n must be either even or odd). so "n^2-2 is never divisible by 4" is true for all n *	A1*
		(4)
		(4 marks)

Notes

M1: Sets up an algebraic statement in terms of a variable (integer) m or any other variable aside n that engages with divisibility by 4 in some way and can lead to a contradiction. No need for explicit statement of assumption - accept if just a suitable equation is set up. In this case supposing using n odd or n even to form an expression for n^2-2 of the form $4\times\text{integer}\pm\text{non-mulitple of 4}$

A1: Reaches $4m^2-2$ or $4(m^2+m)-1$

dM1: For a complete argument that leads to a contradiction. See scheme. Allow if minor details are omitted as long as the overall argument is clear. Both cases must be considered with a reason for the contradiction given (not just stated not divisible by 4).

A1*: Draws the contradiction to their initial assumption and concludes the statement is true for all n. There must have been a clear assumption at the start that is contradicted, and all working must have been correct. For the assumption be generous with the technicality as long as a relevant assumption has been made. E.g. Accept "Assume n^2-2 is divisible be 4 for all n"

Q16.

Question Number	Scheme	Marks

a	b	c	(abc)
6	1	3	(18)
4	2	4	(32)
2	3	5	(30)

	Any one correct row for $b=1$, $b=2$ or $b=3$. Products do not need to be found for this mark.	B1
	Attempts the product abc for at least 2 valid combinations.	M1
	Finds all three valid combinations with correct products seen and somewhere/shows why this is exhaustive and concludes. *	A1*
		(3 marks)

Numerical approach using the table:

B1: Any one correct row for $b = 1$, $b = 2$ or $b = 3$. Products do not need to be found for this mark.
M1: Attempts the product abc for at least 2 valid combinations.
A1*: Requires:

- All three valid combinations with correct products
- No other combinations shown unless they are crossed out or e.g. have a cross at the end of the row or are discounted in some way
- A (minimal) conclusion e.g. the product of a, b and c is even, hence proven, QED, hence it is even, each product stated as even, etc.

Algebraic/logic approach:

B1: Uses the information to obtain a correct equation connecting a and b e.g. $a + 2b = 8$, $a = 8 - 2b$
M1: States a must be even and considers the product abc in some way
A1*: States e.g. abc is even with a reason e.g. "even × anything is even"

Pure Algebraic approach:

B1: Uses the information to obtain a correct equation connecting a and b e.g. $a + 2b = 8$, $a = 8 - 2b$
M1: $abc = (8-2b)b(b+2)$

Attempts the product of a, b and c in terms of b (or some other letter)

A1*: $abc = 2(4-b)b(b+2)$ which is even, hence proven, QED etc.

Concludes abc is even and makes a (minimal) conclusion. There must be no algebraic errors.

NB using this approach "$abc = -2b^3 + 4b^2 + 16b$ which is even hence proven" is not sufficient – they would need to say e.g. which is even + even + even or factor out the 2.

Q17.

Question Number	Scheme	Marks
	$\begin{pmatrix}-1\\5\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\5\end{pmatrix}=\begin{pmatrix}2\\-2\\-5\end{pmatrix}+\mu\begin{pmatrix}4\\-3\\b\end{pmatrix}\Rightarrow \begin{matrix}-1+2\lambda=2+4\mu\ (1)\\5-\lambda=-2-3\mu\ (2)\\4+5\lambda=-5+\mu b\ (3)\end{matrix}$	
	Uses equations (1) and (2) to find either λ or μ e.g. $(1)+2(2)\Rightarrow \mu=...$ or $3(1)+4(2)\Rightarrow \lambda=...$	M1
	Uses equations (1) and (2) to find both λ and μ	dM1
	$\mu=-\frac{11}{2}$ and $\lambda=-\frac{19}{2}$	A1
	$4+5\lambda=-5+\mu b\Rightarrow 4+5\times-\frac{19}{2}=-5-\frac{11}{2}b$ or $4+5\lambda=-5+7\mu\Rightarrow 4+5\times-\frac{19}{2}=-5-\frac{11}{2}\times 7$	ddM1
	$\Rightarrow 11b=77\Rightarrow b=7$ or obtains $-\frac{87}{2}=-\frac{87}{2}$	A1
	States that when $b=7$, lines intersect or when $b\neq 7$, lines do not intersect Lines are not parallel so when $b\neq 7$ lines are skew. *	A1 Cso
		(6)

	Alternative assuming $b=7$:	
	$\begin{pmatrix}-1\\5\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\5\end{pmatrix}=\begin{pmatrix}2\\-2\\-5\end{pmatrix}+\mu\begin{pmatrix}4\\-3\\7\end{pmatrix}\Rightarrow\begin{matrix}-1+2\lambda=2+4\mu & (1)\\ 5-\lambda=-2-3\mu & (2)\\ 4+5\lambda=-5+7b & (3)\end{matrix}$	
	Uses any 2 equations to find either λ **or** μ	M1
	Uses any 2 equations to find both λ and μ	dM1
	$\mu=-\frac{11}{2}$ and $\lambda=-\frac{19}{2}$	A1
	Checks in the 3rd equation e.g. equation 3: $4+5\left(-\frac{19}{2}\right)=-5+7\left(-\frac{11}{2}\right)=...$ equation 1: $-1+2\left(-\frac{19}{2}\right)=2+4\left(-\frac{11}{2}\right)=...$ equation 2: $5-\left(-\frac{19}{2}\right)=-2-3\left(-\frac{11}{2}\right)=...$	ddM1
	Equation 3: $-\frac{87}{2}$ Equation 1: -20 Equation 2: $\frac{29}{2}$	A1
	States that when $b=7$, lines intersect or when $b\neq 7$, lines do not intersect Lines are not parallel so when $b\neq 7$ lines are skew. *	A1 Cso
		(6 marks)

M1: For attempting to solve equations (1) and (2) to find **either** λ **or** μ

dM1: For attempting to solve equations (1) and (2) to find **both** λ **and** μ **<u>Depends on the first M.</u>**

A1: $\mu=-\frac{11}{2}$ and $\lambda=-\frac{19}{2}$

ddM1: Attempts to solve $4+5\lambda=-5+\mu b$ for their values of λ and μ. Or uses $b=7$ with their λ and μ in an attempt to show equality. **<u>Depends on both previous M's.</u>**

A1: Achieves (without errors) that they will intersect when $b=7$

<u>Note that the previous 3 marks may be scored without explicitly seeing the values of both parameters e.g.</u>

$\mu=-\frac{11}{2}$, (2) → $\lambda=3\mu+7$ → $4+5(3\mu+7)=-5+\mu b$ → $b=7$

A1*:Cso States that when $b=7$, lines intersect and since lines are not parallel it shows that when $b\neq 7$ lines are skew.

<u>Alternative:</u>

M1: Uses $b=7$ and attempts to solve 2 equations to find **either** λ **or** μ

dM1: For attempting to solve 2 equations to find **both** λ **and** μ **<u>Depends on the first M.</u>**

A1: $\mu=-\frac{11}{2}$ and $\lambda=-\frac{19}{2}$

ddM1: Attempts to show that the 3rd equation is true for their values of λ and μ

<u>Depends on both previous M's.</u>

A1: Achieves (without errors) that the 3rd equation gives the same values for (or equivalent)

A1*: Cso States that when $b=7$, lines intersect and since lines are not parallel it shows that when $b\neq 7$ lines are skew.

To score the final mark there must be some statement that the lines intersect (or equivalent e.g. meet at a point, cross, etc.) when $b=7$ or that they do not intersect if $b\neq 7$ **and** that the lines are not parallel which may appear anywhere (reason not needed but may be present) so lines are skew when $b\neq 7$.

Ignore any work attempting to show that the lines are perpendicular or not.

Q18.

Question Number	Scheme	Marks
	$\begin{pmatrix} 2-\lambda \\ 8+2\lambda \\ 10+3\lambda \end{pmatrix} = \begin{pmatrix} -4+5\mu \\ -1+4\mu \\ 2+8\mu \end{pmatrix}$	
	Attempts to solve any two of the three equations	
	Either (1) and (2) $\left.\begin{matrix} 2-\lambda=-4+5\mu \\ 8+2\lambda=-1+4\mu \end{matrix}\right\} \Rightarrow \lambda=-\frac{3}{2}, \mu=\frac{3}{2}$	
	(1) and (3) $\left.\begin{matrix} 2-\lambda=-4+5\mu \\ 10+3\lambda=2+8\mu \end{matrix}\right\} \Rightarrow \lambda=\frac{8}{23}, \mu=\frac{26}{23}$	M1, A1
	(2) and (3) $\left.\begin{matrix} 8+2\lambda=-1+4\mu \\ 10+3\lambda=2+8\mu \end{matrix}\right\} \Rightarrow \lambda=-10, \mu=-\frac{11}{4}$	
	Substitutes their values of λ and μ into both sides of the "third" equation	dM1
	E.g. $\lambda=-\frac{3}{2}$ into $10+3\lambda=\frac{11}{2}$ and $\mu=\frac{3}{2}$ into $2+8\mu=14$	
	Concludes that lines don't intersect with correct calculations and minimal reason	A1
	Additionally states that $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 5 \\ 4 \\ 8 \end{pmatrix}$ with a minimal reason	A1*
	So lines are skew CSO *	
		(5) (5 marks)

Notes:
Main method seen
M1: Attempts to solve two of the three equations.
Accept as an attempt, writing down two of the three equations (condoning slips) followed by values for both λ and μ

A1: Solves two of the three equations to find correct values for both λ and μ, Allow equivalent fractions

dM1: Either: Substitutes their values of λ and μ into both sides of the third equation....or into the equations of both lines to find both coordinates

A1: Having achieved correct values for λ and μ, the values for the third equation are found to enable a comparison to be made. E.g. solving equations (1) and (2) and using equation (3) stating $10+3\times-\frac{3}{2}\neq 2+8\times\frac{3}{2}$ is sufficient. If the values are found they must be correct.

Important: Additionally, to score this mark, a minimal statement must be made that states that the lines do not intersect /cross. Condone statements such as $l_1 \neq l_2$
Stating that the lines are skew at this point is not sufficient to score this mark
In the alternative stating that "as the values are not the same, the lines cannot intersect" is sufficient.

A1*: CSO . Hence all previous marks must have been scored.
In addition to not intersecting there must be a statement, with a minimal reason, that the lines are not parallel and hence skew. Accept statements like, not intersecting, not parallel (with reason), hence proven.

Reasons could be $\begin{pmatrix}5\\4\\8\end{pmatrix} \neq k\begin{pmatrix}-1\\2\\3\end{pmatrix}$ o.e such as $5 = -5\times -1$ but $4 = 2\times 2$ so they are not parallel.

Accept an argument based around the scalar product of the direction vectors. If parallel $\cos\theta = 1$

A reason for the lines not being parallel cannot be $\begin{pmatrix}5\\4\\8\end{pmatrix} \neq \begin{pmatrix}-1\\2\\3\end{pmatrix}$

Note: Other methods are possible and it is important that you look at their complete attempt at proving that they don't intersect.
Alternative 1
For example it is possible to solve equations (1) and (2) to find just λ
then solve equations (1) and (3) to find just λ
and then conclude that "as the two values are not the same, the lines don't intersect"

M1 dM1 marks are scored together. Both aspects have to be attempted
Attempts to solve two of the three equations to find λ (or μ)
Attempts to solve a different pair of equations to find λ (or μ)
A1: Correct values for λ (or μ).
A1: conclude that "as the two values are not the same, the lines don't intersect"

If you see something that you feel deserves credit AND that you cannot mark, then please send to review

Q19.

Question	Scheme	Marks
(i)	E.g. $n = 1:\ 2^3 - 1^3 = 7$, $n = 2: 3^3 - 2^3 = 19$, $n = 3:\ 4^3 - 3^3 = \ldots$ Or identifies counterexample directly.	M1
	e.g. $6^3 - 5^3 = 91 = 7\times 13$ so not true for $n = 5$, hence statement is not true.	A1
		(2)
	Notes for part (i) **M1:** Shows evidence of trying to find a counter example for a positive integer (at least one attempt). $2^3 - 1^3$ is prime is sufficient. **A1:** Gives a correct counter example with reason (shows factorisation) and concludes e.g. "which is not prime". Ignore any previous "incorrect" attempts e.g. $6^3 - 5^3 = 91$ which is prime. Note $n = 7$ ($169 = 13\times13$) and $n = 8$ ($217 = 7\times31$) and $n = 12$ ($469 = 7\times67$) are the next few counter examples. (Bigger examples are not likely to be seen!) Allow equivalent reasons for not being prime e.g. 169/13 = 13 or 169 is divisible by 13 (condone "can be divided by 13") **Generally algebraic approaches score no marks unless they substitute numbers as indicated above.**	

(ii)	**The majority of methods here will follow ways 1, 2 or 3 below** **In these cases the general guidance is as follows:** M1: Attempts to find • the gradient of any relevant line, e.g. AC or BC **or** • the length of any relevant line, e.g. AB/AB^2 or BC/BC^2 or AC/AC^2 **or** • the mid-point M of line AB A1: Correct **relevant** calculation of • gradients AC and BC • lengths of lines AB/AB^2, BC/BC^2 and AC/AC^2 • mid-point of line AB dM1: Full attempt at combining all **relevant** information required to solve the problem • attempts product of gradients or equivalent • attempts to show Pythagoras $AB^2 = AC^2 + BC^2$ • attempts to show $MA^2 = MC^2$ A1: Correct calculations or equivalent providing required evidence for the above A1: Provides correct reason and conclusion with all previous marks scored.	

Way 1	$m_{AC} = \frac{-6-0}{7-1} = \ldots$ or $m_{BC} = \frac{-6-(-10)}{7-3} = \ldots$	**M1**
	$m_{AC} = -1$ and $m_{BC} = 1$	**A1**
	So $m_{AC} \times m_{BC} = -1 \times 1 = -1$ or e.g. m_{AC} is negative reciprocal of m_{BC}	**dM1A1**
	So e.g. angle (at *C*) is a right angle hence *AB* is a diameter (Or equivalent)	**A1**
		(5)

M1: Attempts the gradients of *AC* or *BC*. Allow slips but score M0 if both attempts are clearly incorrect.
A1: Correct gradients from correct formulae
dM1: Applies perpendicular condition. May be seen as shown but allow equivalent work.
A1: Correct calculations or equivalent
A1: Suitable explanation and conclusion given with no errors and all previous marks awarded with no incorrect statements seen.

Way 2	$AB = \sqrt{(3-1)^2 + (-10-0)^2} = \ldots$ or $AC = \sqrt{(7-1)^2 + (-6-0)^2} = \ldots$ or $BC = \sqrt{(7-3)^2 + (-6+10)^2} = \ldots$	**M1**
	$AB = \sqrt{104}\left(2\sqrt{26}\right), AC = \sqrt{72}\left(6\sqrt{2}\right), BC = \sqrt{32}\left(4\sqrt{2}\right)$	**A1**
	$AB^2 = 104 = 72 + 32 = AC^2 + BC^2$	**dM1A1**
	Hence *ABC* is a right-angle triangle with hypotenuse *AB* hence *AB* is a diameter. (Or equivalent)	**A1**
		(5)

M1: Attempts length of *AB* or *AC* or *BC* or their squares. Allow slips but score M0 if attempts are clearly incorrect.
A1: Correct values for *AB*, *AC* and *BC* or their squares.
dM1: Applies Pythagoras' theorem with their values. (May see cosine rule used.)
A1: All calculations correct for this approach.
A1: Suitable explanation and conclusion given with no errors and all previous marks awarded with no incorrect statements seen.

Way 3	If *AB* is diameter centre must be midpoint of *AB* ie $M\left(\frac{1+3}{2}, \frac{0-10}{2}\right)$	**M1**
	$=(2,-5)$	**A1**
	$MA = \sqrt{(2-1)^2 + (-5-0)^2} = \sqrt{26}$, $MC = \sqrt{(2-7)^2 + (-5-(-6))^2} = \sqrt{26}$	**dM1A1**
	$MA = \sqrt{26}, MC = \sqrt{26}$ so $MA = MC (= MB)$ As the length from *M* to each of *A* and *C* is the same *M* is the centre of the circle hence *AB* is a diameter. (Or equivalent)	**A1**
		(5)

M1: Attempts midpoint of *AB*. If no method is shown accept one correct coordinate as evidence.
A1: Correct midpoint
dM1: Attempts length of *MC* and at least one of *MA* or *MB*, or *AB*. *As M is midpoint of AB there is no need to find both MA and MB, these may be assumed to be the same.* If they find *AB* then they must halve it to find the radius.
A1: All required calculations correct for this approach.
A1: Suitable explanation made which may be in a preamble and conclusion given with no errors and all previous marks awarded.

The following approach is less common and should be marked as shown:

Way 4	If AB is diameter centre must be midpoint of AB ie $M\left(\frac{1+3}{2},\frac{0-10}{2}\right)$	**M1**
	$=(2,-5)$	**A1**
	$MA=r=\sqrt{(2-1)^2+(-5-0)^2}=\sqrt{26}$ $\Rightarrow(x-2)^2+(y+5)^2=26$ $C(7,-6)\Rightarrow(7-2)^2+(-6+5)^2=5^2+1^2=26$	**dM1A1**
	As C also satisfies the equation of the circle then AB must be the diameter (or equivalent) There must be some further justification as above rather than just "AB is a diameter" which may be in a preamble e.g. If C lies on the circle…	**A1**
		(5)

M1: Attempts midpoint of AB. If no method is shown accept one correct coordinate as evidence.
A1: Correct midpoint
dM1: Attempts length of MA or MB to find r or r^2, forms equation of the circle and substitutes the coordinates of C.
A1: All required calculations correct for this approach.
A1: Suitable explanation and conclusion given with no errors and all previous marks awarded.

There may be other methods. Choose the way that best fits the overall response.
If you are in any doubt if a particular response deserves credit then use Review.

Via perpendicular bisectors:

Way 5	If AB is diameter centre must be midpoint of AB ie $M\left(\frac{1+3}{2},\frac{0-10}{2}\right)$	**M1**
	$=(2,-5)$	**A1**
	Attempts 2 of: $m_{BC}=\frac{-6+10}{7-3}=1$ and midpoint is $\left(\frac{7+3}{2},\frac{-6-10}{2}\right)=(5,-8)$ so perpendicular bisector is $y+"8"=-\frac{1}{"1"}(x-"5")$ or $m_{AC}=\frac{7-1}{-6-0}=-1$ and midpoint is $\left(\frac{7+1}{2},\frac{-6}{2}\right)=(4,-3)$ so perpendicular bisector is $y+"3"=-\frac{1}{"-1"}(x-"4")$ or $m_{AB}=\frac{-10-0}{3-1}=-5$ and midpoint is $\left(\frac{3+1}{2},\frac{-10}{2}\right)=(2,-5)$ so perpendicular bisector is $y+"5"=-\frac{1}{"-5"}(x-"2")$ $y+8=-(x-5)$ oe or $y+3=x-4$ oe or $y+5=\frac{1}{5}(x-2)$ oe **And solves simultaneously:** E.g. $y+3=x-4, y+8=5-x\Rightarrow 5-x-5=x-4\Rightarrow x=2, y=-5$	**dM1**
	Hence centre of circle is $(2,-5)$	**A1**
	E.g. Midpoint of AB is the centre of the circle so AB is a diameter (or equivalent)	**A1**
		(5)

M1: Attempts midpoint of AB. If no method is shown accept one correct coordinate as evidence. **A1:** Correct midpoint **dM1:** Attempts 2 perpendicular bisectors, and solves simultaneously **A1:** Obtains (2, −5) **A1:** Suitable explanation and conclusion given with no errors and all previous marks awarded.

Via circle equation:

Way 6	Uses $(x-a)^2+(y-b)^2=r^2$ With (1, 0), (7, −6) and (3, −10) To find $(a, b) = \ldots$ or $r/r^2 = \ldots$	**M1**
	Centre (2, −5) or radius $\sqrt{26}$	**A1**
	E.g. Equation of AB is $y=-5(x-1)$ and $-5(2-1)=-5$ or $AB=\sqrt{(3-1)^2+(-10-0)^2}=\sqrt{104}=2\sqrt{26}$ or midpoint of AB is $\left(\frac{1+3}{2},\frac{0-10}{2}\right)=(2,-5)$	**dM1A1**
	So centre is on AB or AB is twice the radius or midpoint is the centre hence AB is a diameter of the circle. (or equivalent)	**A1**
		(5)

M1: Uses all three points in circle equation to set up three equations in three unknowns to find centre or radius.
A1: Correct centre or correct radius
dM1: Finds e.g. equation of AB, distance AB or midpoint of AB
A1: Correct equation of AB, distance AB or midpoint of AB
A1: Suitable explanation and conclusion given with no errors and all previous marks awarded.

Via intersecting circles:

(ii) **Way 7**	If AB is diameter centre must be midpoint of AB ie $M\left(\frac{1+3}{2},\frac{0-10}{2}\right)$	**M1**
	$=(2,-5)$	**A1**
	Circle centre C radius r is $(x-7)^2+(y+6)^2=r^2$ Circle centre B radius r is $(x-3)^2+(y+10)^2=r^2$ These intersect when $(x-7)^2+(y+6)^2=(x-3)^2+(y+10)^2$ $\Rightarrow x+y=-3$ Circle centre A radius r is $(x-1)^2+y^2=r^2$ $(x-3)^2+(y+10)^2=(x-1)^2+y^2 \Rightarrow x-5y=27$	**dM1A1**
	Solves simultaneously: $x+y=-3,\ x-5y=27 \Rightarrow x=2, y=-5$	
	E.g. Midpoint of AB is the centre of the circle so AB is a diameter (or equivalent)	**A1**
		(5)

M1: Attempts midpoint of AB. If no method is shown accept one correct coordinate as evidence.
A1: Correct midpoint
dM1: Attempts equations of 2 circles with A, B or C as centre with radius r, repeats the process for 2 different circles and finds the intersection of both straight lines and solves simultaneously
A1: Correct coordinates of centre
A1: Suitable explanation and conclusion given with no errors and all previous marks awarded.

Q20.

Question Number	Scheme	Marks
(i)	States $(S=)\ a+\ (a+d)\ +\dots\dots\{a+(n-2)d\}+\{a+(n-1)d\}$	B1
	$(S=)\{a+(n-1)d\}+\{a+(n-2)d\}+\dots\dots(a+d)\ +a$ and adds	M1
	$2S=n(2a+(n-1)d)\Rightarrow S=\frac{n}{2}\{2a+(n-1)d\}$ *	A1*
		(3)
(ii)	(a) $u_5=22$	B1
	(b) $\sum_{n=1}^{59}u_n=(5+10+15+\dots)+(-3+3-3+\dots)$	
	$=\frac{59}{2}\{10+58\times5\}+(-3)=8850-3=8847$	M1 B1 A1
		(4)
		(7 marks)

(i)

B1 Writes down an expression for S in a minimum of 3 in a and d terms including the first and last terms. Eg. States that $S=a+(a+d)+(a+2d)+\dots\dots+a+(n-2)d+a+(n-1)d$

$S=a+(a+d)+\dots+l$ scores B1 only if $l=a+(n-1)d$ is later identified as only two terms in a and d.

M1 Attempts to reverse their sum and add terms. Must include at least two pairs of matching terms to be enough to establish the pattern (allow if second sum misses last terms).

A1* Correctly achieves the given result including the intermediate line $2S=n\{2a+(n-1)d\}$ There must be no errors and at least 3 terms should have been shown for the sum and its reverse.

If $S=a_1+a_2+\dots+a_n$ is used allow the B and final A only if $a_m=a+(m-1)d$ (oe) is clearly identified in the working, or other clear reasoning why each term gives $2a+(n-1)d$, but the M may be gained.

If commas used instead of + in the summation, eg $S=a,(a+d)\dots,a+(n-2)d,a+(n-1)d$ the score B0 as no correct sum, but allow M1A1 if the sum is implied by working and all else is correct.

If you see other attempts you feel are worthy of credit then consult your team leader.

(ii) (a)

B1 22

(ii) (b)

M1 Attempts to use the sum of an AP with $n=59, a=5, d=5$ Also allow $\frac{n}{2}(a+l)=\frac{59}{2}(5+59\times5)$

B1 For $\sum_{n=1}^{59}3\times(-1)^n=-3$

A1 8847

Listing terms can score 3/3 as it is a correct method and can be marked per main scheme.

Answers with no incorrect working can score 3/3 If the correct answer appears from incorrect working then apply SC M0B1A0.

There are other variations on how to do this part (b)(ii) such as
Alt 1: Splits to odd and even terms:

$$S_{odd} + S_{even} = \frac{1}{2}(2+292)\times 30 + \frac{1}{2}(13+293)\times 29 = 4410 + 4437 = 8847$$

M1 separates into the two sequences and applies summation formula to at least one
B1 Correct expression for both summations (must have correct number of terms for each)
A1 8847

Alt 2: Pairs terms (or can be first term + pairs)

$$S = (2+13)+(12+23)+\ldots+(282+293)+292$$
$$= 15+35+\ldots+575+292 = \frac{29}{2}(15+575)+292 = 8847$$

M1 pairs terms appropriately, may use 1^{st} + (2^{nd}+3^{rd}) + …+ (58^{th} +59^{th}) and applies summation formula to the terms

B1 For the extra term correctly shown 2+… or …+ 292
A1 8847

In general apply
M1 For a correct overall strategy that includes a summation
B1 For dealing with the $\sum(-1)^n$ correctly within the strategy (which may be for a correct overall expression in many cases).
A1 8847

Q21.

Question Number	Scheme	Notes	Marks
(a)	$S_n = a + ar + ... + ar^{n-1}$ $rS_n = ar + ar^2 + ... + ar^n$ Writes down at least 3 correct terms of a geometric series and multiplies their sequence by r. There may be extra incorrect terms but allow this mark if there are 3 correct terms in both sequences and at least one "+" in both sequences **but see special case below**		M1
	$S_n - rS_n = a - ar^n$ or $rS_n - S_n = ar^n - a$ Obtains either equation where both S_n and rSn had the correct first and last terms and at least one other correct term but no incorrect terms. Both sides must be seen unfactorised.		A1(M1 on EPEN)
	$(1-r)S_n = a(1-r^n) \Rightarrow S_n = \frac{a(1-r^n)}{1-r}$ * Factorises both sides and divides by $1 - r$ to obtain the printed answer Should be as printed but allow e.g. $S_n = \frac{a(1-r^n)}{(1-r)}$ but not $S_n = \frac{a(r^n-1)}{(r-1)}$ unless followed by correct version		A1*
	Special case: If terms are listed rather than added and the working is otherwise correct score 110 **See next page for proof by induction.**		
			(3)
	Alternative for (a):		
	$S_n = a + ar + ... + ar^{n-1}$ $(1-r)S_n = (1-r)(a + ar + ... + ar^{n-1})$ or $S_n = \frac{(1-r)(a + ar + ... + ar^{n-1})}{(1-r)}$ Writes down at least 3 correct terms of a geometric series and multiplies both sides by $1 - r$ or multiplies the right hand side by $\frac{1-r}{1-r}$ There may be extra incorrect terms but allow this mark if there are 3 correct terms		M1
	$(1-r)S_n = a - ar^n$ or $S_n = \frac{a - ar^n}{1-r}$ Obtains the above equation where S_n had the correct first and last terms and at least one other correct term and no incorrect terms. Right hand side must be seen unfactorised unless the "a" was factored out earlier		A1 (M1 on EPEN)
	$(1-r)S_n = a - ar^n = a(1-r^n) \Rightarrow S_n = \frac{a(1-r^n)}{1-r}$ * or $S_n = \frac{a - ar^n}{1-r} \Rightarrow S_n = \frac{a(1-r^n)}{1-r}$ * Should be as printed but allow e.g. $S_n = \frac{a(1-r^n)}{(1-r)}$ but not $S_n = \frac{a(r^n-1)}{(r-1)}$ unless followed by correct version		A1*

(b)	**Mark (b) and (c) together**		
	$r^3 = -\frac{20.48}{320} \Rightarrow r = \sqrt[3]{-\frac{20.48}{320}}$	Correct strategy for r. Allow for dividing the 2 given terms **either way round** and attempting to cube root.	M1
	$= -0.4$	Correct value (and no others) but allow equivalents e.g. -2/5. Correct answer only scores both marks.	A1
	Note that some candidates take $ar^2 = -320$ and $ar^5 = \frac{512}{25}$ and use these correctly to give $r^3 = -\frac{20.48}{320} \Rightarrow r = \sqrt[3]{-\frac{20.48}{320}} = -0.4$ **In such cases you can allow full marks for (b) but see note * in (c)**		
			(2)
(c)	$r = -0.4 \Rightarrow a = \frac{-320}{-0.4} (= 800)$ or $r = -0.4 \Rightarrow a = \frac{512}{25} \div \left(-\frac{2}{5}\right)^4 (= 800)$	Correct attempt at the first term using $\pm$ **their** r and the -320 or the $\frac{512}{25}$. May be implied by their a but must be using e.g. $ar = -320$ or $ar^4 = \frac{512}{25}$ **not** $ar^2 = -320$ or $ar^5 = \frac{512}{25}$ *	M1
	$S_{13} = \frac{\text{"}800\text{"}\left(1 - \text{"}-0.4\text{"}^{13}\right)}{1 - \text{"}-0.4\text{"}}$ Correct attempt at the sum using their a and their r and n = 13 to find a value for S_{13}. **Must be a fully correct attempt at the sum here using n = 13, their a and their r.** Note that $\frac{800\left(1+0.4^{13}\right)}{1+0.4}$ is equivalent to $\frac{800\left(1-(-0.4)^{13}\right)}{1-(-0.4)}$ and is acceptable for this mark.		M1
	$= 571.43$	Correct value. Note that S_∞ is also 571.43 so working must be seen i.e. correct answer only scores no marks.	A1
			(3)
			Total 8

Proof by induction for part (a):

$$n = 1 \Rightarrow S_1 = \frac{a\left(1-r^1\right)}{1-r} = a \quad \text{so true for } n = 1$$

$$\text{Assume true for } n = k \text{ so } S_k = \frac{a\left(1-r^k\right)}{1-r}$$

$$\text{Add } (k+1)^{th} \text{ term } S_{k+1} = \frac{a\left(1-r^k\right)}{1-r} + ar^k = \frac{1 - ar^k + ar^k - ar^{k+1}}{1-r}$$

$$= \frac{a - ar^{k+1}}{1-r} = \frac{a\left(1-r^{k+1}\right)}{1-r}$$

So if true for $n = k$ it has been shown true for $n = k + 1$ and as it is true for $n = 1$ it is true for (for all n)

Mark as follows:

M1: Shows true for $n = 1$ and assumes true for $n = k$ and adds the $(k + 1)^{\text{th}}$ term

A1(M1 on EPEN): Finds common denominator obtains $\frac{a - ar^{k+1}}{1-r}$ using correct algebra

A1: Fully correct proof reaching $\frac{a\left(1-r^{k+1}\right)}{1-r}$ with all steps shown and conclusion

If you are in any doubt about awarding marks in this case or any other cases that you think deserve credit, send to your Team Leader using Review

Q22.

Question Number	Scheme	Marks
(a)	$x=2, y=5 \Rightarrow 5=8a-12+6+b$	M1
	$\frac{dy}{dx}=3ax^2-6x+3$ **AND** $x=2, \frac{dy}{dx}=7 \Rightarrow 7=12a-12+3$	M1
	Solves $11=8a+b$ and $7=12a-9 \Rightarrow a=\frac{4}{3}, b=\frac{1}{3}$	A1 A1
		(4)
(b)	Sets $\frac{dy}{dx}=3ax^2-6x+3=0$ with their value of a and b	M1
	$4x^2-6x+3=0$ and attempts "b^2-4ac"	dM1
	$b^2-4ac=-12<0$ hence there are no turning points oe	A1*
		(3)
		(7 marks)

(a)

M1 Substitutes $x=2, y=5$ into $y=ax^3-3x^2+3x+b$ to get an equation in a and b (condone slips)

M1 Substitutes $x=2, \frac{dy}{dx}=7$ into $\frac{dy}{dx}=3ax^2-6x+3$ to get an equation in a ($\frac{dy}{dx}$ must be correct)

A1 $a=\frac{4}{3}$ or exact equivalent

A1 $b=\frac{1}{3}$ or exact equivalent

(b)

M1 Sets their $\frac{dy}{dx}=0$ with their value of a. This may be implied by later working.

dM1 Attempts to find b^2-4ac or roots via the formula

A1* Achieves $4x^2-6x+3=0$, $b^2-4ac=-12$ and states $-12<0$ and so there are *no turning points* or equivalent. They may attempt to solve the equation and either state that *no real roots so no turning points* or achieve complex roots and state that *no real roots so no turning points*. Note full marks can be scored with an incorrect value for $b=\frac{1}{3}$ but cannot be scored from an incorrect value for $a=\frac{4}{3}$ from part (a)

Alt (b)

M1 Attempts to complete the square for $\frac{dy}{dx}=3ax^2-6x+3$ for their value of a to achieve eg $4(x\pm...)^2$ or $(2x\pm...)^2$

dM1 $4x^2-6x+3=4\left(x\pm\frac{3}{4}\right)^2\pm...$ or $4x^2-6x+3=\left(2x\pm\frac{3}{2}\right)^2\pm...$

A1* Achieves $4x^2-6x+3=4\left(x-\frac{3}{4}\right)^2+\frac{3}{4}$ or $4x^2-6x+3=\left(2x-\frac{3}{2}\right)^2+\frac{3}{4}$ and states there are *no turning points* as $\frac{dy}{dx}>0$ (for all x) or equivalent.

If you see any other ways that may be credit worthy then send to review