Matrix Transformations: Further Maths

matrix : an array of information, presented in rows and columns.

- The dimentions of the matrix are presented in the form: $a \times b$, where 'a' is the number of rows and 'b' is the number of columns.

$$\begin{array}{c} \text{1 column} \\ \begin{pmatrix} a \\ b \end{pmatrix} \end{array} \text{2 rows}$$

2 × 1 matrix

- Each number in the matrix is called an element. The number of elements in a matrix is the number of rows multiplied by the number of columns. There are no gaps in matrices; if there isn't any data in place of an element - then a 0 is written.

Multiplying matrices

→ Multiplying by a scalar

scalar: an integer, constant value.

- To multiply a matrix by a scalar quantity:

1. Multiply each element of the matrix by the scalar value.

eg "$4\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$"

$= \begin{pmatrix} 4\times4 & 4\times-1 \\ 4\times-3 & 4\times2 \end{pmatrix} = \begin{pmatrix} 16 & -4 \\ -12 & 8 \end{pmatrix}$

→ Multiplying by a matrix

- If the matrices multiplication is : $A \times B = C$

→ number of columns in A = number of rows in B

→ C will have the same number of rows as A, and same number of columns as B.

→ If A and B are square, you can also find $B \times A$ - but this isn't usually equal to $A \times B$.

TIP! Remember to multiply matrices in the correct order as $A \times B$ rarely equals $B \times A$.

- To multiply two matrices:

1. Multiply the first column of A by the first element in B (top-left) in its first column.
2. Multiply the second column of A by the second element in B (bottom-left) in its first column.
3. Add the values to get the first column of the matrix products.
4. Repeat the process using matrix A but with the second column of B, if the matrix is bigger.

eg 1 2 × 2 matrix multiplied by a 2 × 1 matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2+0 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

eg 2 2×2 matrix multiplied by a 2×2 matrix

$$\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0+4 & 0+4 \\ 0+6 & 1+6 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 6 & 7 \end{pmatrix}$$

Identity and zero matrices

→ The identity matrix

- If you multiply matrix, A, by the identity matrix, you get A.
- The 1×1 identity matrix is 1.
- The 2×2 identity matrix is: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It is often called 'I' or 'I_2' in questions.

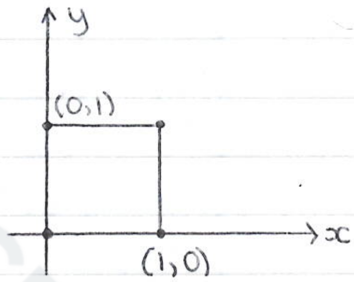
→ The zero matrix

- If you multiply matrix, A, by the zero matrix, you get the zero matrix. This is because anything multiplied by 0 equals 0.
- The zero matrix is a matrix of zeros. The 2×2 zero matrix is: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Transformations of the unit square

- A matrix can represent any point on a coordinate grid. $(a, b) = \begin{pmatrix} a \\ b \end{pmatrix}$
- Matrices can represent some transformations (rotations, reflections & enlargements) and can transform coordinates on a grid.
- To transform a coordinate by a matrix:
  1. Convert the coordinate into a 2×1 matrix.
  2. Multiply the matrix by the 2×1 matrix to get the transformed coordinate.
- When we transform a point, we call the new location of the point the image of the original point under the transformation matrix it was multiplied by. We say the original point (x, y) is mapped to (x', y').

→ The unit square: a square drawn in the 1^{st} quadrant of a set of axis, where each dimention is 1 unit.



- Transforming the points (1,0) and (0,1) of the unit square can help work out what a transformation matrix represents.
- To work out what a matrix represents:
  1. Multiply (1,0) and (0,1) of the unit square by the transformation matrix to locate their image.
  2. Draw the unit square and their image points to visualise the transformation.
  3. Describe the transformation in full detail.
- To work out the matrix that performs a specific transformation:
  1. Draw the unit square and transform the points (1,0) and (0,1) by the specific

transformation to get their image.

2. The image of (1,0) gives the first column of the transformation matrix. The image of (0,1) gives the second column.

Transformation matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

→ $\begin{pmatrix} a \\ c \end{pmatrix}$ is the image of (1,0)

→ $\begin{pmatrix} b \\ d \end{pmatrix}$ is the image of (0,1)

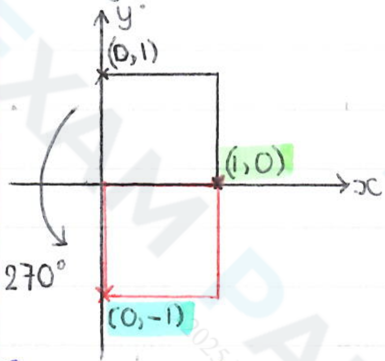
TIP! The transformation matrix looks like the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ before the image points are substituted in.

eg "Find the matrix that represents the transformation of rotation through 270°, about the origin."

unless told assume anticlockwise

▦ = unit square

▦ = image



The image of (1,0) is (0,−1).

The image of (0,1) is (1,0).

Transformation matrix: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Combining Transformations

- Combining transformations (or composite transformations) is when you combine two matrix transformations into one.
- If a point is transformed by both transformation matrices separately, it will give the same image as the same point transformed by the combined matrix.
- To combine transformation matrices:

1. Multiply the transformation matrices together in the order that they happen.

eg "Point A (2,3) is transformed by M $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and then by R $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to give point B. What are the coordinates of point B?"

$A = (2,3) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$B = MRA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$MR = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0\times1 + -1\times0 & 0\times0 + -1\times-1 \\ 1\times1 + 0\times0 & 1\times0 + -1\times0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\times0 + 1\times3 \\ 2\times1 + 0\times3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = (3,2)$

What were the transformations?

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ → rotation, 90°, origin

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ → reflection x-axis

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ → reflection y = x