Calculus: Further Maths

Differentiation

→ Formula to differentiate powers of x: $\frac{d}{dx}(x^n) = nx^{n-1}$

→ Notation:

- The thing inside the brackets is what is being differentiated. If $y = x^n$, then write: dy/dx. $\frac{Dy}{dx}$ is pronounced "dy by dx."
- The gradient function dy/dx gives the gradient of the curve and measures the rate of change of y with respect to x ($\therefore$ in terms of x).
- Alternate notation:
  → If $y = ax^n$, $dy/dx = anx^{n-1}$.
  → If $f(x) = ax^n$, $f'(x) = anx^{n-1}$.

→ Differentiation rules:

- Multiply the power by the coefficient of x, then take away 1 from the power of x.

eg $2x^3 \rightarrow 6x^2$

$\frac{d}{dx}(2x^3) = (3 \times 2)x^{3-1} = 6x^2$

- If there is a coefficient of x^1 it differentiates to just the coefficient without the x.

eg $2x \rightarrow 2$

$\frac{d}{dx}(2x) = 2(x^{1-1}) = 2 \times x^0 = 2 \times 1 = 2$

- Constants differentiate to 0.

eg $-7 \rightarrow 0$

let $-7 = -7x^0$

$\frac{d}{dx}(-7x^0) = (-7 \times 0)(x^{0-1}) = 0 \times x^{-1} = 0$

- Differentiate each term in the equation separately.
- Rearrange nasty expressions into powers of x to make them easier to differentiate.

eg 1 "$y = \frac{(\sqrt{x})^6 + (x-2)^2 - 4}{x}$. Work out dy/dx."

$y = \frac{(x^{1/2})^6 + (x-2)^2 - 4}{x} = \frac{x^3 + (x-2)^2 - 4}{x}$ (use power rules)

$= \frac{x^3 + x^2 - 4x}{x} = x^2 + x - 4$ (expand & cancel)

$dy/dx = 2x + 1$

eg 2 "Given $y = \frac{5}{x^3}$. Work out dy/dx."

$y = \frac{5}{x^3}$

$y = 5\left(\frac{1}{x^3}\right)$

$y = 5x^{-3}$

$dy/dx = (-3 \times 5)(x^{-3-1}) = -15 \times x^{-4} = \frac{-15}{x^4}$

→ Gradient

- The gradient of a curve is constantly changing. You can draw a tangent to estimate the gradient at a point, but differentiation can find it exactly.
- Differentiating the equation of a curve gives an expression for the curve's gradient.
- To find the gradient of the curve at a certain point:

1. Differentiate the equation to get the gradient expression.
2. Substitute the x-value into the expression.

eg "Find the gradient at the point (2,16) on the graph $y = 3x^4$."

$dy/dx = 12x^3$

gradient $= 12(2)^3 = 96$

- Differentiating the equation measures the rate of change (how fast something is increasing/decreasing compared to something else.) "The rate of change of y with respect to x" means how fast y is changing compared to x. It is the same as the gradient.

Tangents and normals

→ Tangents : a straight line which touches the curve at a single point.

- To find the equation of a tangent to the curve:

1. Differentiate the equation.
2. Substitute the x-value into the derivative to find the gradient.
3. → Write the equation in the form $y - y_1 = m(x - x_1)$.
   → Write the equation in the form $y = mx + c$: substitute in the values into the equation $y = mx + c$ & rearrange to find 'c'.

TIP! If one of the values in the coordinate where the tangent meets the curve is missing: substitute the known value into the original equation to find the missing one.

eg 1 "Find the equation of the tangent to the curve $y = 3x^2 - 5x - 2$ at the point $(1, -4)$."

$dy/dx = 6x - 5$

tangent gradient at point $(1,-4) = 6(1) - 5 = 1$

tangent equation:

→ $y - y_1 = m(x - x_1)$ → $y + 4 = x - 1$

→ $y = mx + c$ → $y = x - 5$

$-4 = 1(1) + c$ (-1)

$-5 = c$

eg 2 "Find the equation of the tangent to the curve $y = x^3 - 9x^2 + 23x - 15$ where $x = 2$."

Find the y-value: $y = (2)^3 - 9(2)^2 + 23(2) - 15 = 3$ $\therefore (2, 3)$

$dy/dx = 3x^2 - 18x + 23$

tangent gradient at point $(2,3) = 3(2)^2 - 18(2) + 23 = -1$

tangent equation:

→ $y - y_1 = m(x - x_1)$ → $y - 3 = -1(x - 2)$

→ $y = mx + c$ → $y = 5 - x$

$3 = -1(2) + c$

$5 = c$ $(+2)$

→ Normals: a straight line that is perpendicular to a tangent.

• To find the equation of a normal of the curve:

1. Differentiate the equation.
2. Find the gradient of the tangent.
3. Find the gradient of the normal by doing the negative reciprocal of the tangent's gradient.
4. → Write the equation in the form $y - y_1 = m(x - x_1)$

→ Write the equation in the form $y = mx + c$: substitute in the values into the equation $y = mx + c$ & rearrange to get 'c'.

eg 1 "Find the equation of the normal to the curve $y = 3x^2 - 5x - 2$ at the point $(1, -4)$."

$dy/dx = 6x - 5$

tangent gradient at point $(1, -4) = 6(1) - 5 = 1$

normal gradient $= -1$

normal equation:

→ $y - y_1 = m(x - x_1)$ → $y + 4 = -1(x - 1)$

→ $y = mx + c$ → $y = -x - 3$

$-4 = -1(1) + c$

$-3 = c$ $(+1)$

Stationary points

- Stationary points are where the gradient of the curve becomes 0.
  $\therefore dy/dx = 0$. They can be a minimum, maximum or 'point of inflection' (where the graph goes flat for a bit).
- To find the stationary points of a graph:
  1. Differentiate the equation to find dy/dx.
  2. Set dy/dx equal to 0 and solve for x.
  3. Substitute the x-value into the original equation to find the y-values of the turning points.

eg "For the curve $x^3 - 12x + 3$ find the coordinates of the turning points."

$dy/dx = 3x^2 - 12$

At turning points, $dy/dx = 0$.

$3x^2 - 12 = 0$

$3x^2 = 12$ $\quad (+12)$

$x^2 = 4$ $\quad (\div 3)$

$x = \pm 2$ $\quad (\sqrt{\ })$

when $x = 2$

$y = (2)^3 - 12(2) + 3 = -13$

when $x = -2$

$y = (-2)^3 - 12(-2) + 3 = 19$

$\therefore$ turning points at $(2, -13)$ and $(-2, 19)$.

second derivative method !

- How to check if a stationary point is maximum or minimum:
  1. Differentiate dy/dx again to get the second derivative d^2y/dx^2.
     → d^2y/dx^2 is the rate of change of gradient
     → d^2y/dx^2 is positive when the gradient is increasing and negative when it is decreasing.
     → The gradient increases through a minimum & decreases through a maximum.
  2. Substitute the x-value of the stationary point into d^2y/dx^2 to see if it's positive or negative.
     → If $d^2y/dx^2 > 0$, the turning point is a minimum.
     → If $d^2y/dx^2 < 0$, the turning point is a maximum.

eg "The stationary points of the curve $y = f(x)$ are at $(-1, -1/3)$ and $(3, -11)$. Given that $dy/dx = x^2 - 2x - 3$, determine the nature of each stationary point."

$d^2y/dx^2 = 2x - 2$

$x = -1 \rightarrow d^2y/dx^2 = 2(-1) - 2 = -2 - 2 = -4 < 0$, so $(-1, -1/3)$ is a maximum.

$x = 3 \rightarrow d^2y/dx^2 = 2(3) - 2 = 6 - 2 = 4 > 0$, so $(3, -11)$ is a minimum.

- To use gradient to find the type of stationary point:
  - → Positive gradient on the left & negative on the right: maximum.
  - → Negative gradient on the left & positive on the right: minimum.
  - → The same gradient either side: point of inflection.

eg "The curve $2x^4 - 9x^2 + 10$ has a stationary point at $(1.5, -1/8)$. Determine the nature of this stationary point, giving reasons for your answer."

$dy/dx = 8x^3 - 18x$

$x = 1 \Rightarrow dy/dx = 8(1)^3 - 18(1) = 8 - 18 = -10 < 0$

$x = 2 \Rightarrow dy/dx = 8(2)^3 - 18(2) = 64 - 36 = 28 > 0$

The gradient is negative on the left & positive on the right, so $(1.5, -1/8)$ is a minimum.

- To find out if a function is increasing/decreasing at a point in an interval:
  1. Differentiate the equation to find the curve's gradient.
  2. Substitute the x-value into the derivative for the gradient at that point.
     - → If the gradient is positive, then that interval is increasing.
     - → If the gradient is negative, then that interval is decreasing.

eg "For the following points is the function $y = x^2 + 3x - 9$ increasing or decreasing:"

a) "$(4, 19)$"

$dy/dx = 2x + 3$

$x = 4 \Rightarrow dy/dx = 2(4) + 3 = 8 + 3 = 11 > 0$, so the function is increasing.

b) "$(-2, -11)$"

$x = -2 \Rightarrow dy/dx = 2(-2) + 3 = -4 + 3 = -1 < 0$, so the function is decreasing.

- To find the intervals of a function increasing/decreasing:
  1. Differentiate the equation for the curve's gradient.
  2. Solve dy/dx equal to 0 to find the critical points (turning points).
  3. Choose two x-values either side of the critical point & substitute them into dy/dx. Use the results to determine the increasing/decreasing intervals.
     - → If the gradient is positive, the interval is increasing.
     - → If the gradient is negative, the interval is decreasing.

eg 1 "Determine the intervals in which the function $f(x) = x^2 + 2x$ is increasing/decreasing."

$f'(x) = 2x + 2$

$2x + 2 = 0$

$2 = -2x \quad (-2x)$

$-1 = x \quad (\div -2)$ $\quad \therefore -1$ is the only critical value & the x-value for the turning point

Two values for x either side of the critical value: 0 & -2.

$x = 0 \rightarrow f'(0) = 2(0) + 2 = 2 > 0$, so the function is increasing.

$x = -2 \rightarrow f'(-2) = 2(-2) + 2 = -4 + 2 = -2 < 0$, so the function is decreasing.

Intervals:

$x < -1$, decreasing

$x > -1$, increasing

eg 2 "For what values of x is $y = 2x^2 + 12x - 7$ a decreasing function?"

$dy/dx = 4x + 12$

For a decreasing function: $4x + 12 < 0$

$$4x < -12 \quad (-12)$$

$$x < -3 \quad (\div 4)$$

eg 3 "Use differentiation to show that $f(x)$ is an increasing function for all values of x. $f(x) = 2x^3 - 12x^2 + 25x - 11$."

$$\begin{aligned} f'(x) &= 6x^2 - 24x + 25 \\ &= 6[x^2 - 4x] + 25 \\ &= 6[(x-2)^2 - 4] + 25 \\ &= 6(x-2)^2 - 24 + 25 \\ &= 6(x-2)^2 + 1 \end{aligned}$$

$(x-2)^2 \geq 0$ for all values of x (because anything squared is 0 or positive)

$6(x-2)^2 + 1 > 0$ for all values of x (because 0 or anything positive $+1$ is always positive)

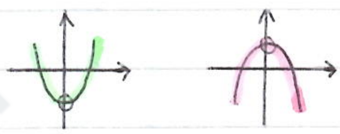
Drawing curves

→ The coefficient of the largest power of x defines the graph's shape:

- Quadratic graphs

a positive coefficient of x^2: a 'U' shape

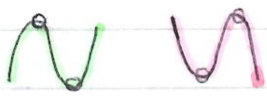
a negative coefficient of x^2: a '∩' shape



- Cubic graphs

a positive coefficient of x^3: curve goes up from bottom left

a negative coefficient of x^3: curve goes down from top left



- Stationary points can help sketching a graph as the type it is tells us what the graph should do either side of that point.

→ Intercepts

- To find x-intercepts: set $y = 0$ into the original equation to find the x-values or solve the equation.

- To find the y-intercept : set $x = 0$ into the original equation to find the y-value or look at the constant in the equation for its y-value.