Algebra : Further Maths

Indices

→ Writing numbers "as a power of" another number: use knowledge of square, cubic etc. numbers. (e.g. "8 as a power of 2" = 2^3)

→ Solving equations with indices: by using indices laws & rewriting parts as a power of another number.

eg 1 $x^{\frac{3}{2}} = 8$

$(\sqrt{x})^3 = 8$ (indices laws)

$\sqrt{x} = 2$ ($\sqrt[3]{\ }$)

$x = 4$ (^{2})

eg 2 $2^{3x} \times 8^2 = 4^{2x}$

$2^{3x} \times 2^6 = (2^2)^{2x}$

$2^{3x+6} = 2^{4x}$

$\therefore\ 3x+6 = 4x$ ($-3x$)

$6 = x$

eg 3 $x^{\frac{-1}{2}} = \frac{1}{3}$

$\sqrt{\frac{1}{x}} = \frac{1}{3}$ (indices laws)

$\frac{\sqrt{1}}{\sqrt{x}} = \frac{1}{3}$

$\frac{1}{\sqrt{x}} = \frac{1}{3}$

$\frac{1}{x} = \frac{1}{9}$ (^{2})

$\therefore\ x = 9$

alternatively

$\frac{1}{x} = \frac{1}{9}$ ($\times x$)

$1 = \frac{1}{9}x$

$\frac{1}{\frac{1}{9}} = x$ ($\div \frac{1}{9}$)

$\frac{9}{1} = 9 = x$ (due to the fractional rule)

$\boxed{\text{Fractional rule: } \frac{1}{\frac{a}{b}} = \frac{b}{a}}$

→ Solving disguised quadratics with indices:

1. Ensure the equation is equal to 0.
2. Let a letter (e.g 'y') equal an amount of the unknown so it forms a quadratic equation.
3. Solve for the letter to get 2 solutions, then replace the letter with the amount of the unknown & solve for solutions.

general TIP! Check you've found the correct amount of solutions (2 for quadratic, 3 for cubic, 4 for quartic etc.) depending on the original equation & if they are legitimate.

eg1 $x^4 = 5x^2 - 4$

$x^4 - 5x^2 + 4 = 0$

$y^2 - 5y + 4 = 0$ let $y = x^2$

$(y-4)(y-1)=0$

$\therefore\ y=4 \quad \text{or} \quad y=1$

$x^2=4 \qquad x^2=1 \qquad$ (sub in $y=x^2$)

$x=\pm 2 \qquad x=\pm 1 \qquad (\sqrt{\ })$

eg2 $\frac{3}{\sqrt{x}}+\frac{2}{x}=5$

$\frac{3}{\sqrt{x}}+\frac{2}{x}-5=0$

$\frac{3}{y}+\frac{2}{y^2}-5=0 \qquad$ let $y=\sqrt{x}$

$(\times y^2)$

$-5y^2+3y+2=0 \quad \Rightarrow$ solve

eg3 $4^x=2^x+56$

$(2^2)^x=2^x+56$

$2^{2x}-2^x-56=0$

$y^2-y-56=0 \qquad$ let $y=2^x$

$(y-8)(y+7)=0$

$\therefore\ y=8 \quad \text{or} \quad y=-7$

$2^x=8 \qquad 2^x=-7 \qquad$ (sub in $y=2^x$)

$x=3 \qquad$ n/a

eg4 $9^x=3^x+6$

$(3^2)^x=3^x+6$

$3^{2x}-3^x-6=0$

$y^2-y-6=0 \qquad$ let $y=3^x \qquad \Rightarrow$ solve

eg5 $x^{\frac{1}{3}}-2x^{-\frac{1}{3}}-1=0$

$x^{\frac{1}{3}}-2\left(\frac{1}{x^{\frac{1}{3}}}\right)-1=0 \qquad$ (indices laws)

$y-2\left(\frac{1}{y}\right)-1=0 \qquad$ let $y=x^{\frac{1}{3}}$

$y^2-y-2=0 \qquad (\times y) \quad \Rightarrow$ solve

eg6 $4^x-2^{x+1}-48=0$

$2^{2x}-2(2^x)-48=0 \longrightarrow 2^{x+1}=2^x\times 2^1=2(2^x)$

$y^2-2y-48=0 \qquad$ let $y=2^x \qquad \Rightarrow$ solve

Expanding brackets review

→ single brackets: multiply everything in the bracket by the term outside the bracket. Be careful × check if the term is positive or negative.

→ double brackets: multiply everything in the 1st bracket by everything in the 2nd.

→ triple brackets: expand 2 brackets first × put it in brackets, then multiply it with the 3rd bracket.

→ squared brackets: expand them to avoid confusion → $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$ NOT $a^2 + b^2$.

Binomial Expansion

→ Brackets to the power of 0 equals to 1 due to indices laws.

→ Brackets with higher powers can take long to expand (e.g $(x+1)^5$) so we use another method.

→ Pascal's triangle gives the co-efficients of the powers of the terms.

row number:	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 6 1

- there are 1s on the outside of the triangle
- the 2nd number corresponds to the row number
- the power outside the bracket shows which row to use to get the coefficients of the powers of the terms.
- Pascal's triangle works by having 1s on the outside & adding the two numbers above it to get the next number in the row.

(it is symetrical)

$\boxed{a \quad b \to a+b \quad \text{eg } 1\ 3\ 3\ 1 \to 1\ 4\ 6\ 4\ 1}$

- Calculator method for finding a row of Pascal's triangle for higher powers:

nCr ← choose button 'C'
÷ ← shift & press

1. Use the choose button by pressing shift & then the division button.
2. Enter: x 'C' y
   - 'x' = the number of the row of Pascal's triangle wanted.
   - 'y' = start at 0, then adding 1 each time until you get to 'x' to get the whole row. $x \geq y$

e.g row 4
4C0 = 1
4C1 = 4
4C2 = 6
4C3 = 4
4C4 = 1

→ Powers of the terms

- The power of the bracket is what the first & last terms have the power of. It's also the number of powers that the terms' powers should sum to in each 'section'.
- As the powers of the 1st term in the bracket to be expanded decreases, the powers of the 2nd increases - and vica versa.

TIP! • If the power is even, the term will always be positive. (eg $-2^2 = 4$ $2^2 = 4$)
If the power is odd, the term can be either positive or negative. (eg $-2^3 = -8$ $2^3 = 8$)

eg

$$(x+y)^5 = (1)(x)^5(y)^0 + (5)(x)^4(y)^1 + (10)(x)^3(y)^2 + (10)(x)^2(y)^3 + (5)(x)^1(y)^4 + (1)(y)^5(x)^0$$

5 5 5 5 5 5

→ anything to the power of 0 = 1

row 5 co-efficients:
1, 5, 10, 10, 5, 1

- To write the terms "in descending powers of" something, expand that term first.

eg "Expand $(1+2x)^8$ for the first 3 terms of the expansion, in descending powers of x."

$(1+2x)^8 = (1)(2x)^8 + (8)(2x)^7(1)^1 + (28)(2x)^6(1)^2 + \dots$

$= 256x^8 + 1024x^7 + 1792x^6 + \dots$

$+ \dots$ shows the expansion will continue

$8C0 = 1$
$8C1 = 8$
$8C2 = 28$

- To write the terms "in ascending powers of" something, expand the other term first.

"Expand $(1+x)^{10}$ for the first 3 terms of the expansion, in ascending powers of x."

eg $(1+x)^{10} = (1)(1)^{10} + (10)(1)^9(x)^1 + (45)(1)^8(x)^2 + \dots$

$= 1 + 10x + 45x^2 + \dots$

$10C0 = 1$
$10C1 = 10$
$10C2 = 45$

→ Surds can also be in the brackets of a binomial expansion.

eg $(2+\sqrt{5})^3 = (1)(2)^3 + (3)(2)^2(\sqrt{5})^1 + (3)(2)^1(\sqrt{5})^2 + (1)(\sqrt{5})^3$

$= 8 + 12\sqrt{5} + 30 + 5\sqrt{5}$

$= 38 + 17\sqrt{5}$

$3C0 = 1$
$3C1 = 3$
$3C2 = 3$
$3C3 = 1$

Factorising Hidden DOTS

They could be:

→ In a fraction or needs to be further factorised:

eg $\frac{x^2-81}{2x+18} = \frac{(x+9)(x-9)}{2(x+9)} = \frac{x-9}{2}$

→ Using higher powers of x. (TIP! Any even power is another power squared.)

eg $x^{10} - 4y^2 = (x^5+2y)(x^5-2y)$ because $x^{10} = (x^5)^2$

→ You might have to use DOTS to find a common factor when simplifying.

eg $(m^2-49) + (m+7)(2m+3) = (m+7)(m-7) + (m+7)(2m+3)$

$= (m+7)[(m-7) + (2m+3)]$

$= (m+7)(3m-4)$

Factor Theorem: factorising polynomials

→ If $f(x)$ is a polynomial, and $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$.

PROOF! $f(x) = x^2 - 8x + 12 = (x-6)(x-2) \xrightarrow{\text{solve}} x = 6$ & $x = 2$

sub in $x = 6$ $\quad f(6) = (6-6)(6-2) = 0 \times 4 = 0$ ✓ $\therefore (x-6)$ is a factor of $f(x)$

sub in $x = 2$ $\quad f(2) = (2-6)(2-2) = -4 \times 0 = 0$ ✓ $\therefore (x-2)$ is a factor of $f(x)$

→ It can also be used if the factor is in the form $(ax-b)$.

eg If $f(0.5) = 0$, then $(2x-1)$ is a factor → $2x - 1 = 0$

$x - 0.5 = 0$ $(\div 2)$

$x = 0.5$ $(+0.5)$

eg2 "Show that $(x-1)$ is a factor of $f(x)$. $f(x) = x^3 + 3x^2 - x - 3$."

$f(1) = 0$ ↲ substitue in

$f(1) = 1^3 + 3(1)^2 - 1 - 3 = 1 + 3 - 1 - 3 = 0$ ✓

statement: $f(x) = 0$ when $x = 1$, so by the Factor Theorem, $(x-1)$ is a factor of $f(x)$.

→ To factorise a cubic equation, 3 factors are needed, which can be linear in the form of $(ax-b)$, $(x-b)$ or a term on its own (e.g $\underline{2}(\cdot\cdot)(\cdot\cdot)$ or $\underline{x}(\cdot\cdot)(\cdot\cdot)$).

→ Trial & error method:

- Try substituting in 1: if all co-efficients sum to 0, then $(x-1)$ is a factor (e.g $2x^3 + 7x^2 - 4x - 5 \rightarrow 2 + 7 - 4 - 5 = 0$)
- Try substituting small numbers & factors of parts of the equation like: $0, \pm 1, \pm 2, \pm 3$.

→ Long division method & remainder theorem

1. Find 1. factor of the equation.
2. Put the equation in a division box with the factor on the outside.
3. Look at the first part of the factor ('x') & work out what you'd need to multiply it by to get the first term of the equation. Put this on top of the division box.
4. Multiply the term on top by each term of the factor outside the box & put it below the equation below the equation in brackets.
5. Subtract the terms below from the equation.
6. Repeat steps 3-5 with the product of each subtraction until you reach 0. The terms on top of the division is what you're left with after dividing the equation by the factor so you can easily factorise the rest.

$$
\begin{array}{r|l}
 & x^2 + 4x + 3 \\
\hline
x-1 & x^3 + 3x^2 - x - 3 \\
- & (x^3 - x^2) \\
\hline
 & \quad 4x^2 - x - 3 \\
- & \quad (4x^2 - 4x) \\
\hline
 & \qquad 3x - 3 \\
- & \qquad (3x - 3) \\
\hline
 & \qquad\qquad 0
\end{array}
$$

TIP! If it is a factor, the end will always $= 0$.

When finding the remainder after a division, the end will not always equal 0. e.g →

"Find the remainder when $x^3 - 5x^2 + 3x + 8$ is divided by $(x-1)$."

$$
\begin{array}{r|l}
 & x^2 - 4x \\
\hline
x-1 & x^3 - 5x^2 + 3x + 8 \\
- & (x^3 - x^2) \\
\hline
 & \quad -4x^2 + 3x + 8 \\
- & \quad (-4x^2 + 4x) \\
\hline
 & \qquad -x + 8 \\
- & \qquad (-x + 8) \\
\hline
 & \qquad\qquad 7
\end{array}
$$

Simultaneous equations

→ To solve simultaneous equations with 3 unknowns:

1. There will be at least 3 equations. Label the equations ①, ② & ③.
2. Eliminate the same variable from 2 of the 3 equations, using 2 different equations each time. This creates 2 new equations. Label them ④ & ⑤.
3. Solve ④ & ⑤ like a normal simultaneous equation with 2 unknowns.
4. Substitute the 2 found variables into one of the first 3 equations to find the third unknown.

eg

$3x + 2y - 3z = -13$ ①
$2x - 3y + 4z = 24$ ②
$4x - 5y + 2z = 22$ ③

② × 3, ① × 2 (subtract):
$6x - 9y + 12z = 72$
$6x + 4y - 6z = -26$
$-13y + 18z = 98$ ④

② × 2, ③ (subtract):
$4x - 6y + 8z = 48$
$4x - 5y + 2z = 22$
$-y + 6z = 26$ ⑤

sub y into ④:
$-13(6z - 26) + 18z = 98$
$-60z = 240$
$\underline{z = 4}$

sub z=4 into ⑤:
$-y + 6(4) = 26$
$-y = 2$
$\underline{y = -2}$

sub in z & y into ①:
$3x + 2(-2) - 3(4) = -13$
$3x - 16 = -13$
$3x = 3$
$\underline{x = 1}$

eg 2 "The first 3 terms of a quadratic sequence with n^{th} term $= an^2 + bn + c$ are 0, 3 & 10. Find the values of a, b & c."
1^{st} term ⇒ $a + b + c = 0$ ① $[a(1)^2 + b(1) + c = a + b + c$ which are constants$]$
2^{nd} term ⇒ $4a + 2b + c = 3$ ② $[a(2)^2 + b(2) + c = 4a + 2b + c]$
3^{rd} term ⇒ $9a + 3b + c = 10$ ③ $[a(3)^2 + b(3) + c = 9a + 3b + c]$
→ then solve with the 3 equations

4 unknowns:

→ To solve simultaneous equations with 4 or more unknowns:
1. Eliminate the same variable from three equations.
2. Solve like a simultaneous equation with 3 unknowns.
- It is similar for more unknowns.

Limiting value of a sequence

→ When a sequence has a limiting value, they get to a certain value & then each term remains constant. Linear and quadratic sequences don't have a limiting value.

→ To find the limiting value of a sequence as "$n \to \infty$"

"→" is read as: tends to

1. Divide every term by the highest power of n.
2. As n gets bigger, $\frac{x}{n}$ gets smaller & closer to 0. Hence the fractions are seen as 0.
3. The remainder of the sequence is what the squence tends to.

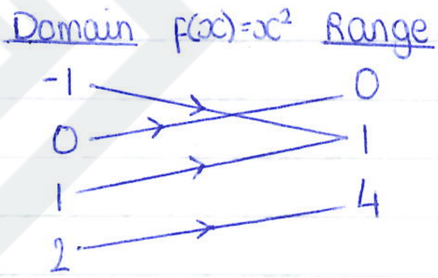
eg "Find the limiting value of the sequence $\frac{2n-1}{3n+2}$ as $n \to \infty$."

$$\frac{2n-1}{3n+2} \ (\div n) = \frac{2-\frac{1}{n}}{3+\frac{2}{n}}$$

as $n \to \infty$, $\frac{1}{\infty} \approx 0 \therefore \frac{1}{n} \approx 0$, $\frac{2}{n} \approx 0$

as $n \to \infty$, $\frac{2-\frac{1}{n}}{3+\frac{2}{n}} \to \frac{2-0}{3+0} = \frac{2}{3}$

Functions



function : a mathematical relationship that maps each value of a set of inputs (domain) to a single output (range). Each domain value maps to only one range value, but different domain values can map to the same value in the range.

- The notation $f(x)$ is used to represent a function of x.
- A function is another way of writing an equation. For example: instead of writing an equation like "$y = 5x+2$", you can write a function like "$f(x) = 5x+2$" or "$f: x \to 5x+2$".
- The roots of a function are the values of x for which $f(x) = 0$.

→ Domain & Range

Domain: the set of possible inputs for a function. They are given in terms of x and often described with inequalities.

Range: the set of possible outputs for a function. They are given in terms of $f(x)$.

- To find the domain of a function:

You are trying to find all of the values that 'work' with the function. Sometimes "the domain is all real values of x", but other times there are values that need to be excluded. Consider these things when trying to find the domain:

→ The denominator of a fraction cannot be zero.

→ You can't find the square or even root of a negative number (so the number underneath must be positive or zero).

The domain can be written using words, inequalities or/and with the union notation ('U'). If the input of a function, x, can be any real number: the domain can be written as "$x \in \mathbb{R}$".

→ The symbol "$\in$" means "is a member of"/"is an element of".

→ The symbol "$\mathbb{R}$" represents the real numbers.

examples :

$f(x) = \sqrt{x}$ The domain is $x \geqslant 0$

$f(x) = \frac{1}{x}$ The domain is $x \neq 0$

$f(x) = x^2$ The domain is all real values of x. ($x \in \mathbb{R}$)

$f(x) = \frac{1}{5-x}$ The domain is $x<5 \cup x>5$ / $x \neq 5$ / all real values of x, except 5 etc.

• To find the range of a function:

Method 1 (for basic functions) :

1. Substitute in values from the domain into the function.
2. Substitute in other values to see if the function is increasing or decreasing.
3. Use this information & common sense to work out the range.

eg $f(x) = x+4$ Domain: $x>1$ Range: $x>1$

$f(1) = 1+4 = 5$

$f(2) = 2+4 = 6$

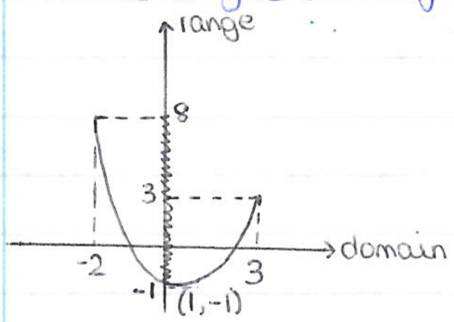
increasing function: as the input increases, the output increases.

(It's better to draw a graph though)

Method 2 (for harder functions) :

1. Draw the graph of the function.
2. Find the corresponding y-values of the domain values & any turning points.
3. Draw long, vertical lines from the domain values up to their corresponding y-values & horizontal lines from any turning points.
4. Colour on the y-axis from the highest to lowest y-values between the domain values to give the range.

eg



$f(-2) = (-2)^2 - 2(-2) = 4+4 = 8$

$f(3) = (3)^2 - 2(3) = 9-6 = 3$

TP : $x = 1$ (symmetry)

$f(1) = 1-2 = -1 = y$ $\Big\}$ $(1,-1)$

$f(x) = x^2 + 2x$ Domain: $-2 \leqslant x \leqslant 3$ Range: $-1 \leqslant f(x) \leqslant 8$

→ Increasing & decreasing functions

• An increasing function is when the y-value increases as the x-value increases. This is shown on a graph where the gradient is positive. If there isn't a graph, algebra can be used to see if the function is increasing. For a function $y=f(x)$:

→ Increasing: when $x_1 < x_2$, then $f(x)_1 \leq f(x)_2$

→ Strictly increasing: when $x_1 < x_2$, then $f(x)_1 < f(x)_2$

eg $f(x) = 2x + 5$

$x_1 = 2 \rightarrow f(2) = 2(2)+5 = 9$

$x_2 = 3 \rightarrow f(3) = 2(3)+5 = 11$

$9<11$ so it is strictly increasing

• A decreasing function is when the y-value decreases as the x-value increases. This is shown on a graph where the gradient is negative. For a function $y=f(x)$:

→ Decreasing: when $x_1 < x_2$, then $f(x)_1 \geq f(x)_2$

→ Strictly decreasing: when $x_1 < x_2$, then $f(x)_1 > f(x)_2$

eg $f(x) = 5 - x$

$x_1 = 2 \rightarrow f(2) = 5-2 = 3$

$x_2 = 3 \rightarrow f(3) = 5-3 = 2$

$3>2$ so it is strictly decreasing

• The function is constant when the gradient is 0 (the graph is flat).

• The graph of the function may not be increasing/decreasing/constant everywhere (e.g. $f(x) = x^2$), so sometimes we are only interested in the interval where this happens.

→ To find x, given $f(x)$: substitute the value in place of the $f(x)$, then solve like an equation.

eg "$f(x) = \frac{1-2x}{x+4}$. Given that $f(x) = -0.5$, find x."

$-0.5 = \frac{1-2x}{x+4}$

$-1(x+4) = 2(1-2x)$ $\quad (\times 2)(\times x+4)$

$-x-4 = 2-4x$

$6 = 3x$ $\quad (\times -1)$ $(-x)(+2)$

$2 = x$

→ Composite functions

composite function: combination of 2 or more functions; evaluating the input of a function & inputting that answer into another function.

• written like $fg(x)$ (pronounced: "f of g of x")

• To evaluate a composite function: input from the inside & work outwards from x.

$fg(x)$

$f \quad g \quad (x)$

① evaluate $g(x)$

② evaluate answer in $f(x)$

"Find $fg(2)$. $f(x) = 5x + 2$ $g(x) = 2x + 1$."

$g(2) = 2(2) + 1 = 4 + 1 = 5$

$f(5) = 5(5) + 2 = 25 + 2 = 27$ $(fg(2) = 27)$

- To write a composite function as an expression:

1. Rewrite the composite function but expanded with more brackets.
2. Sub in the inner most function.
3. Sub in the outer function like evaluating (like $2x + 1 = x$) → expand + simplify.

"Expression for $fg(x)$."

1. $f(g(x))$
2. $f(2x+1)$
3. $5(2x+1) + 2$
   $= 10x + 5 + 2 = 10x + 7$

- To express a composite function as the expressions corresponding to the individual functions: break down the function into as many parts as there are individual functions.

e.g. "Express $(5x+6)^4$ in the form $fg(x)$, stating the expressions corresponding to $f(x)$ and $g(x)$."

$f(x) = 5x + 6$

$g(x) = x^4$

→ Inverse functions

inverse function: a function that does the opposite of the original function.

- A function must be a one-to-one to have an inverse function, meaning each value in the range has only one domain value mapping to it.
- written as $f^{-1}(x) = \ldots$ (meaning: inverse function of $f(x)$)
  (pronounced: "f inverse x")

- To find the inverse function of a function:

1. Replace $f(x)$ for y → writing the function as: "$y = \ldots$".
2. Rearrange to make "x" the subject of the equation
3. Swap the xs & ys together.
4. Replace "y" with "$f^{-1}(x)$".

"$f(x) = \frac{12 + x}{3}$. Find $f^{-1}(x)$."

① $y = \frac{12 + x}{3}$

② $3y = 12 + x$ $(\times 3)$

$x = 3y - 12$ (-12)

③ $y = 3x - 12$

④ $f(x) = 3x - 12$

→ Graphs of functions

- To draw the graph of a function:

1. Find the necessary information to draw the graph. Some features include:
   → To find the roots (x-intercepts): solve the function.
   → To find the y-intercept: substitute 0 as the x-value in $f(x)$.
   → To find the turning point: complete the square, use symmetry or differentiation & then solve.

→ To find the shape: look at the equation.

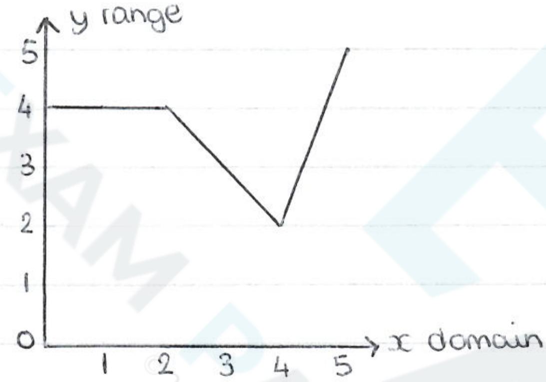
→ To find points on the graph of the function: substitute the domain value as the x-value to get the range as the y-value for the coordinates - and vice versa.

2. Draw the axes: the x-axis is the domain & the y-axis is the range.
3. Draw the graph of the function.

TIP! stop drawing the graph where the domains finish.

- Functions can be defined by different equations for different parts of the domain. You could be asked to draw the graph of a function like this:

eg



"A function $g(x)$ is defined as

$$g(x) = 4 \qquad 0 \leq x < 2$$
$$= 6 - x \qquad 2 \leq x \leq 4$$
$$= 3x - 10 \qquad 4 < x \leq 5$$

Draw the graph $y = g(x)$ for $0 \leq x \leq 5$."

- You could also be asked to find the function & domain by interpreting a graph.

→ To find the function's equation: work out the equation of each straight line of the graph. Each line gives an equation.

→ To find the domain: look at where each line starts & ends on the x-axis. Each line gives a domain for the line's equation.